MS-A0111
FALL 2021

SEQUENCES

$$(a_n)_{n \in \mathbb{N}} = (a_n)_{n=1}^{\infty}$$
$$= (a_1, a_2, a_3, \dots)$$

$\mathbb{N}$ = natural numbers $= \{1, 2, 3, \dots\}$

$a_n \in \mathbb{R}$ ($\mathbb{R}$ real numbers)

A sequence is a function:

$$f : \mathbb{N} \to \mathbb{R}$$

$$f(n) = a_n, \quad n = 1, 2, \dots$$

DEFINITION A sequence (a_n) converges to a limit $\alpha \in \mathbb{R}$, if

$$|a_n - \alpha| \xrightarrow[n \to \infty]{} 0 .$$

Formal: (ε, δ) - version

For every $\varepsilon > 0$ there exists an index $n = n(\varepsilon)$, such that

$$|a_n - \alpha| < \varepsilon \text{ for every } n \geq n(\varepsilon).$$

We write: $\lim_{n \to \infty} a_n = \alpha$.

EXAMPLE $\lim_{n \to \infty} \frac{1}{n^2} = 0$

$$\left| \frac{1}{n^2} - 0 \right| = \frac{1}{n^2} < \varepsilon,$$

if $n > \frac{1}{\sqrt{\varepsilon}}$. $\quad n(\varepsilon) = \lceil 1/\sqrt{\varepsilon} \rceil$.

We can take ε arbitrarily small!

GEOMETRIC SEQUENCE AND SERIES

$$\frac{a_{n+1}}{a_n} = q \quad (= \text{constant})$$

$$\Leftrightarrow \quad a_{n+1} = q\, a_n$$

$$\Rightarrow \quad a_n = a_1 q^{n-1}, \quad q \neq 0, \; q \neq 1$$

Series:

$$S_n = a_1 + a_2 + \dots + a_n$$

$$= a_1 \frac{1 - q^n}{1 - q}, \quad q \neq 1$$

$$S = \sum_{n=1}^{\infty} a_n = a_1 \sum_{n=0}^{\infty} q^n$$

This converges to $\dfrac{a_1}{1-q}$,

if $|q| < 1$,

otherwise it diverges.

EXAMPLE Show that

$$1 + \frac{1}{x} + \frac{1}{x^2} + \dots = \frac{x}{x-1},$$

assuming $x > 1$.

Solution: $x > 1$, $\dfrac{1}{x} < 1$

Here $S = \sum_{n=0}^{\infty} q^n = \dfrac{1}{1-q}$

$$1 + \frac{1}{x} + \frac{1}{x^2} + \dots = \frac{1}{1 - \frac{1}{x}} = \frac{x}{x-1}$$

$$\begin{array}{r|l} x - 1 & x \\ & \underline{x - 1} \\ & 1 \\ & \underline{1 - \frac{1}{x}} \\ & \frac{1}{x} \end{array} \quad = 1 + \frac{1}{x} + \frac{1}{x^2} + \dots$$

FUNCTIONS

Function f is a mapping from a domain to its range.

$$f : A \to B$$

Often

$$f_A = \{ f(a) \mid a \in A \} \subset B$$

↳ image

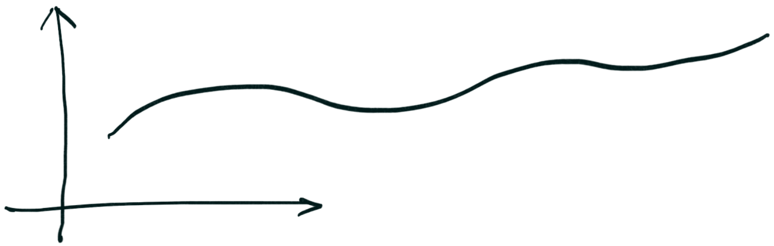
DEFINITION Continuity

If,

Always when $a_n \in A$, and

$\lim\limits_{n \to \infty} a_n = a$, then

$$\lim_{n \to \infty} f(a_n) = f(a).$$



THEOREM

$$\lim_{x \to a} f(x) = A, \quad \lim_{x \to a} g(x) = B$$

$$\Rightarrow \lim_{x \to a} (f(x) + g(x)) = A + B$$

Proof

Let $\varepsilon > 0$. Then $\exists\, \delta_1 > 0, \delta_2 > 0$

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - A| < \frac{\varepsilon}{2}$$

$$0 < |x - a| < \delta_2 \Rightarrow |g(x) - B| < \frac{\varepsilon}{2}$$

choose $\delta_\varepsilon = \min(\delta_1, \delta_2)$,

then

$$0 < |x - a| < \delta_\varepsilon \Rightarrow$$

$$|f(x) + g(x) - (A + B)| =$$

$$|(f(x) - A) + (g(x) - B)| \le \left(\begin{array}{c}\text{triangle}\\\text{ineq.}\end{array}\right)$$

$$|f(x) - A| + |g(x) - B| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$\square$