THEOREM

Let $f(x_0) = 0 = g(x_0)$ and both differentiable in the neighbourhood of x_0. If the limit $\lim_{x \to x_0} \frac{f(x)}{g(x)}$ exists, the

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} .$$

<u>Not a formal proof</u> :

Assume the the derivatives are continuous and $g'(x_0) \neq 0$.

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} \overset{\text{IVT}}{=} \frac{f'(\xi_1)(x - x_0)}{g'(\xi_2)(x - x_0)}$$

$$= \frac{f'(\xi_1)}{g'(\xi_2)} \xrightarrow[x \to x_0]{} \frac{f'(x_0)}{g'(x_0)}$$

DERIVATIVE

Newton's Quotient

DEFINITION

$$\lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

if this limit exists ($\in \mathbb{R}$), then f is differentiable at a, and the limit is the derivative of f at a.

$$f'(a) = Df(a) = \left.\frac{df}{dx}\right|_{x=a}$$

EXAMPLE $\frac{d}{dx} \sin x = \cos x$

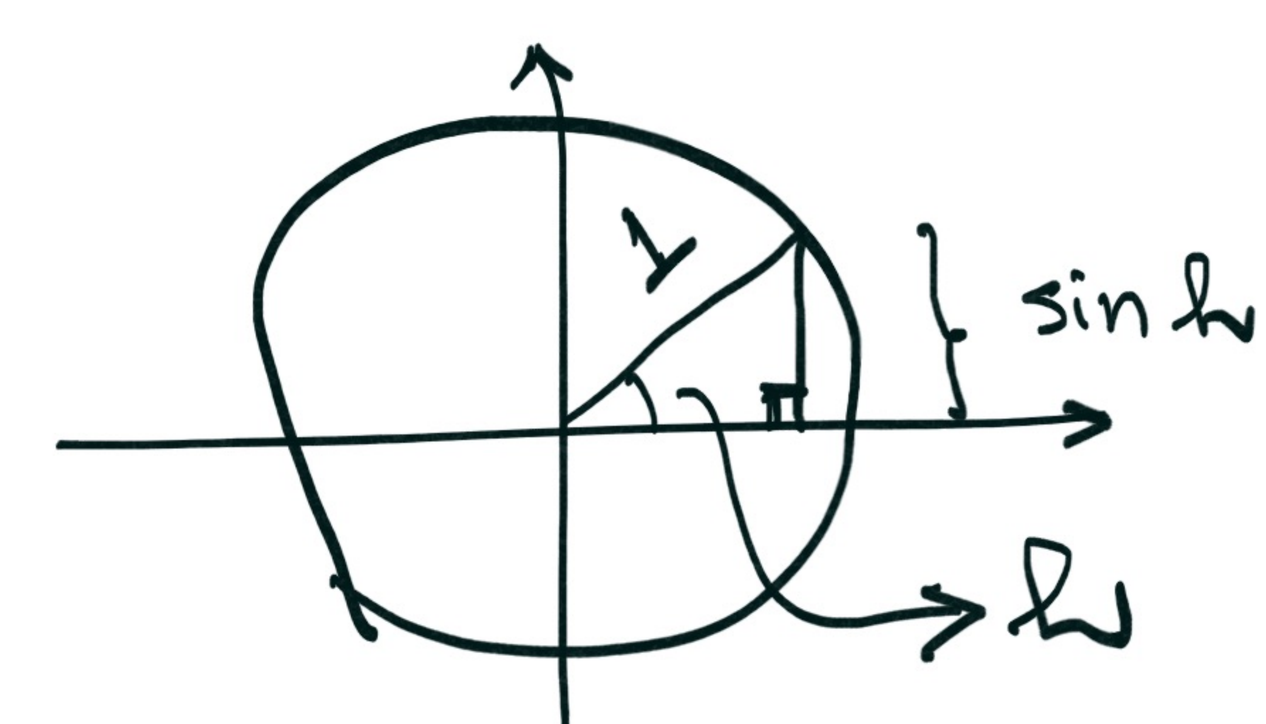
$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \frac{\sin x (\cos h - 1)}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h}$$

$$= \cos x$$



RULES

$$D(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

$$D\,\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}, \quad g(x) \neq 0$$

Why? At point a:

$$y = \frac{f}{g}\ ; \quad \frac{\Delta y}{\Delta x} = \frac{1}{\Delta x}\left[\frac{f(a)+\Delta f}{g(a)+\Delta g} - \frac{f(a)}{g(a)}\right]$$

$$= \frac{g(a)(f(a)+\Delta f) - (g(a)+\Delta g)f(a)}{\Delta x\,(g(a)+\Delta g)\,g(a)}$$

$$= \frac{g(a)\frac{\Delta f}{\Delta x} - f(a)\frac{\Delta g}{\Delta x}}{g(a)(g(a)+\Delta g)}$$

Let $\Delta x \to 0$ and we're done!

$$D\,f(g(x)) = g'(x)\,f'(g(x))$$

$\hookrightarrow$ the chain rule

l'Hospital's Rule

Intermediate Value Theorem:

THEOREM

Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous and differentiable on the open interval (a,b) $(=]a,b[\,)$.

Then there exists a point $\xi \in (a,b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b-a}$$

$$\left(\text{or } f(a) - f(b) = f'(\xi)(b-a)\right)$$