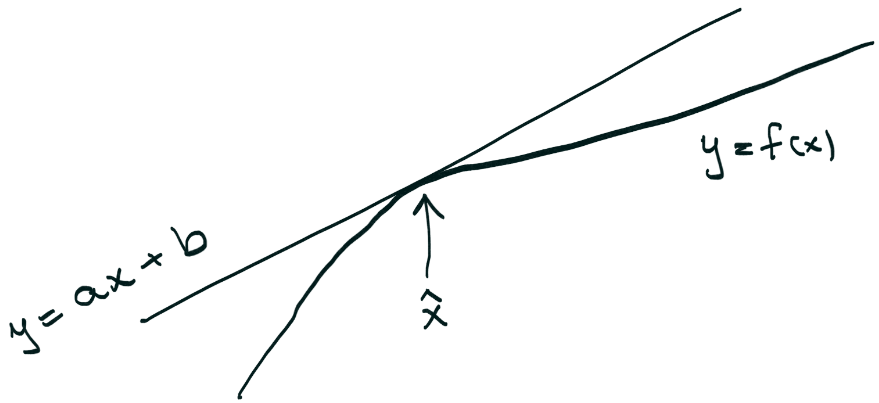
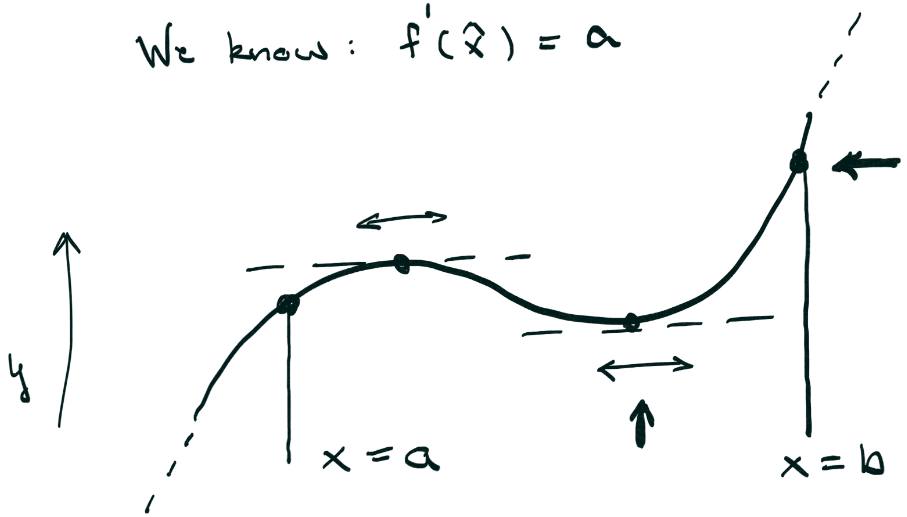
# APPLICATIONS OF DERIVATIVES




We know: $f'(\hat{x}) = a$




## THEOREM

If $f'(x_0)$ exists and is $> 0$ ($< 0$), then $f$ is increasing (decreasing) at $x_0$.

## Weierstrass:

Every range of a continuous function over a closed interval has a maximum and a minimum.

## THEOREM

If $f'(x_0) = 0$ and $f''(x_0) > 0$ ($< 0$) then $f(x_0)$ is a local minimum (maximum).

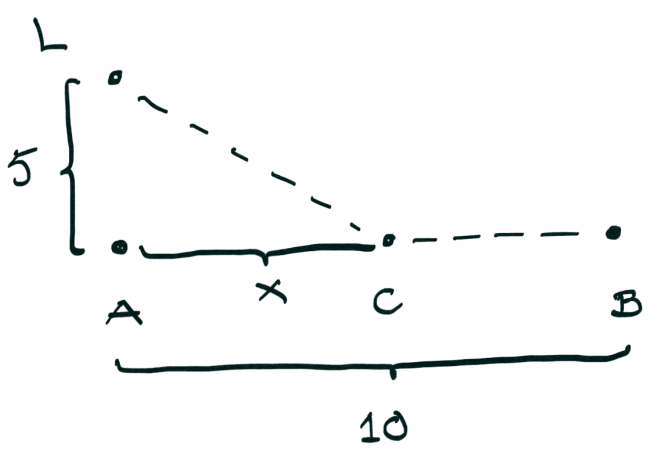
## EXAMPLE

$f(x) = x^2 \Rightarrow f''(x) = 2$

$g(x) = x^3 \Rightarrow g''(x) = 6x$

$\Rightarrow g''(0) = 0$ (indeterminate)

EXAMPLE "CABLE LAYING PROBLEM"




Cost: $BC = 3000$ / unit
$LC = 5000$ / unit

Total cost:

$$T = T(x) = 5000\sqrt{25+x^2} + 3000(10-x)$$

$$\frac{dT}{dx} = \frac{5000x}{\sqrt{25+x^2}} - 3000 = 0$$

$$\Rightarrow x = \frac{15}{4}$$

$T(0) = 55000$

$T(10) \approx 55900$

$T\left(\frac{15}{4}\right) = 50000$

# TRANCENDENTAL FUNCTIONS

Exponential and logarithmic functions:

$y = a^x$, $a \in (0, \infty) \setminus \{1\}$

$y = e^x$, $e$ is the Euler's number.

Notice: $e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$, $x \in \mathbb{R}$

DEFINITION

The inverse of $y = e^x$ is the natural logarithm: $\ln : \mathbb{R}_+ \to \mathbb{R}$

$$y = \ln x \iff x = e^y$$

Rules: $\ln xy = \ln x + \ln y$

$\ln x^y = y \ln x \,, \quad y \in \mathbb{R}$

$$e^{\ln xy} = xy = e^{\ln x} e^{\ln y} = e^{\ln x + \ln y}$$

Derivatives:

$$\frac{de^x}{dx} = e^x \,, \quad \frac{d \ln x}{dx} = \frac{1}{x}$$

Newton's Quotient:

$$\frac{e^{x+h} - e^x}{h} = \frac{e^x (e^h - 1)}{h}$$

DEFINITION $a^x = e^{x \ln a} \,, \quad a > 0$

Derivative:

$$\frac{da^x}{dx} = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \ln a = a^x \ln a$$

Logarithmic function:

$$y = \log_a x \iff x = a^y$$

(Implicit differentiation)

Assume that $y = y(x)$ and take derivatives on both sides:

$$x = a^y \overset{D}{\Longrightarrow} 1 = a^y \ln a \frac{dy}{dx} = x \ln a \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x \ln a} = \frac{d}{dx} \log_a x$$

# Inverse Trigonometrix Functions

Sine : $f(x) = \sin x \, , \; -\pi/2 \leq x \leq \pi/2$

DEFINITION

$y = \arcsin x \iff x = \sin y$

Derivative : $x = \sin y \overset{D}{\Longrightarrow}$

$$1 = \cos y \frac{dy}{dx}$$

$$\cos y = -\sqrt{1 - \sin^2 y} = -\sqrt{1 - x^2}$$

$$\Rightarrow \frac{d \arcsin x}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

Tangent : $f(x) = \tan x \, , \; -\pi/2 < x < \pi/2$

DEFINITION

$y = \arctan x \iff x = \tan y$

Derivative

$$x = \tan y \overset{D}{\Longrightarrow} 1 = \frac{1}{\cos^2 y} \frac{dy}{dx}$$

$$= (1 + \tan^2 y) \frac{dy}{dx}$$

$$= (1 + x^2) \frac{dy}{dx}$$

$$\Rightarrow \frac{d}{dx} \arctan x = \frac{1}{1 + x^2}$$