$$\lim_{x \to 0} \frac{2\left(x - \frac{x^3}{3!}\right) - \left(2x - \frac{(2x)^3}{3!}\right)}{2\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right) - 2 - 2x - x^2}$$

$$= \lim_{x \to 0} \frac{-\frac{x^3}{3} + \frac{4x^3}{3}}{\frac{x^3}{3}} = 3$$

TAYLOR POLYNOMIALS

DEFINITION

The linearisation of the function f about a point a is the function L defined as

$$L(x) \simeq f(a) + f'(a)(x-a)$$

EXAMPLE

$\sqrt{26} \simeq ?$ $\quad f(x) = \sqrt{x}$

Choose $a = 25 \Rightarrow f(a) = 5$

$f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}} \Rightarrow L(x) = 5 + \frac{1}{10}(x-25)$

$\Rightarrow L(26) = 5.1$

ERROR ESTIMATION $\quad (t > a)$

$$E(t) = f(t) - f(a) - f'(a)(t-a)$$

Therefore:

$$E'(t) = f'(t) - f'(a)$$

Generalised Mean Value Theorem:

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

Let us choose $E(t)$ and $(t-a)^2$ on $[a,t]$:

$$\frac{E(t) - \overbrace{E(a)}^{=0}}{(t-a)^2 - \underbrace{(a-a)^2}_{=0}} = \frac{E'(\xi)}{2(\xi - a)} = \frac{1}{2} f''(\eta)$$

$$\Rightarrow E(t) = \frac{1}{2} f''(\eta)(t-a)^2$$

since $E(a) = 0$ (!)

TAYLOR CONSTRUCTION

Key idea: Pointwise approximation

Assumptions:

$f(a), f'(a), \dots, f^{(n-1)}(a)$ exist

Let us write:

$$T_{n-1}(x, a) = c_0 + c_1(x-a) + \dots + c_{n-1}(x-a)^{n-1}$$

$$T'_{n-1}(x, a) = c_1 + 2c_2(x-a) + \dots + (n-1)c_{n-1}(x-a)^{n-2}$$

$\dots$

$$T^{(k)}_{n-1}(x, a) = k!\,c_k + (x-a)\underbrace{P(x)}_{\text{some polynomial}}, \quad k = 1, 2, \dots, n-2$$

$\dots$

$$T^{(n-1)}_{n-1}(x, a) = (n-1)!\,c_{n-1}$$

Condition:

$$T^{(k)}_{n-1}(x, a) = f^{(k)}(a), \quad k = 0, 1, 2, \dots, n-1$$

This leads to a unique set of coefficients c_k:

$$c_k = \frac{f^{(k)}(a)}{k!}, \quad k = 0, 1, \dots, n-1$$

DEFINITION Taylor Polynomial

$$T_{n-1}(x, a) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

THEOREM Lagrange Remainder

If $f^{(n)}(x)$ is continuous over $[a, x]$, then

$$f(x) = T_{n-1}(x, a) + \frac{f^{(n)}(\xi)}{n!}(x-a)^n,$$

where $\xi \in [a, x]$.

EXAMPLE

Maclaurin polynomial, if $a = 0$.

$f(x) = \sin x$ $\quad$ $g(x) = \cos x$

$f'(x) = \cos x$ $\quad$ $g'(x) = -\sin x$

$f''(x) = -\sin x$ $\quad$ $g''(x) = -\cos x$

$f'''(x) = -\cos x$ $\quad$ $g'''(x) = \sin x$

$f^{(4)}(x) = \sin x$ $\quad$ $g^{(n)}(x) = \cos x$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \Theta(x^7)$$

$$= \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \Theta(x^{2n+3})$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \Theta(x^6)$$

$$= \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + \Theta(x^{2n+2})$$

THEOREM (Alternative formulation)

If $f(x) = Q_n(x) + \Theta((x-a)^{n+1})$ as $x \to a$, where $Q_n(x)$ is a polynomial of degree at most n, then $Q_n(x) = T_n(x, a)$.

That is, Q_n is the Taylor polynomial of $f(x)$ about $x = a$.

Application: Limits

EXAMPLE

$$\lim_{x \to 0} \frac{2\sin x - \sin 2x}{2e^x - 2 - 2x - x^2}$$

$$\sin x \simeq x - \frac{x^3}{3!}$$

$$\sin 2x \simeq 2x - \frac{(2x)^3}{3!}$$

$$e^x \simeq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \quad \left(= \sum_{k=0}^{n} \frac{x^k}{k!} \right)$$