The sum function is continuous and differentiable on I.

Therefore

$$f'(x) = \sum_{k=1}^{\infty} k\, c_k (x - x_0)^{k-1}.$$

EXAMPLE

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

$$D \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} (2k+1)\, x^{2k}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \cos x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$D \sin x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$= \cos x$$

Similarly for definite integrals:

$$\int_a^b f(x)\,dx = \sum_{k=0}^{\infty} c_k \int_a^b (x - x_0)^k\,dx$$

SERIES

THEOREM

If the series $\sum_{k=1}^{\infty} a_k$ converges, then

$$\lim_{k \to \infty} a_k = 0.$$

Otherwise, the series diverges.

Notice : Harmonic Series

$$\sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges,}$$

even though $\frac{1}{k} \xrightarrow[k \to \infty]{} 0$.

Rules of summation :

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$$

$$\sum_{k=1}^{\infty} (c a_k) = c \sum_{k=1}^{\infty} a_k$$

$$\left(\sum_{k=1}^{\infty} a_k\right)\left(\sum_{k=1}^{\infty} b_k\right) = ?$$

$$(a_1 + a_2)(b_1 + b_2)$$

$$= a_1 b_1 + a_2 b_2 + a_1 b_2 + a_2 b_1$$

(Cauchy product)

Here we have assumed that the series are convergent.

EXAMPLE

$$\sum_{n=1}^{\infty} \frac{e^n}{n^2} = ?$$

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

(l' Hospital) $\Rightarrow$ series diverges

Convergence Tests (samples)

THEOREM

If $|a_k| \leq p_k$ and $\sum_{k=1}^{\infty} p_k$ converges, then also

$$\sum_{k=1}^{\infty} a_k \text{ converges.}$$

We say that $\sum_{k=1}^{\infty} p_k$ is the majorant.

There is a complementary concept, the minorant; for $0 \leq p_k \leq a_k$ and if $\sum p_k$ diverges then

$$\sum a_k \text{ diverges.}$$

EXAMPLE

$$\frac{1}{\sqrt{k}} \geq \frac{1}{k}, \text{ for all } k \in \mathbb{N},$$

hence $\sum \frac{1}{\sqrt{k}}$ diverges.

THEOREM (Ratio Test)

$\left| \frac{a_{k+1}}{a_k} \right| \leq Q$, starting at some k.

Q: constant $0 < Q < 1$, then

$$\sum a_k \text{ converges.}$$

Proof:

$$|a_k| \leq Q|a_{k-1}| \leq Q^2 |a_{k-2}| \leq \dots \leq Q^k |a_0|$$

$\square$

EXAMPLE

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2} = ?$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{3^{n+1}}{(n+1)^2}\right)}{\left(\frac{3^n}{n^2}\right)}$$

Streamlining the notation:

$$\frac{3^{n+1}}{3^n} \cdot \frac{n^2}{(n+1)^2} \longrightarrow 3\left(\frac{n}{n+1}\right)^2$$

$$\longrightarrow 3\left(1 - \frac{1}{n+1}\right) \xrightarrow{n\to\infty} 3$$

The series diverges.

TAYLOR SERIES

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad x \in \mathbb{R}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad x \in \mathbb{R}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}, \quad x \in \mathbb{R}$$

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k, \quad |x| < 1$$

This cannot be computed for every function, however.

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f^{(k)}(0) = 0 \text{ for all } k.$$

POWER SERIES

DEFINITION

$$\sum_{n=1}^{\infty} c_n (x - x_0)^n = \lim_{m \to \infty} \sum_{k=0}^{m} c_k (x - x_0)^k$$

Terminology: x_0 is the centre

c_k are the coefficients

The series converges at x if the limit exists.

Radius of convergence:

R : The power series converges on some interval $(x_0 - R, x_0 + R)$, where R is the radius of convergence.

For the Taylor series above, $R = \infty$ or $R = 1$, for the pathological example $R = 0$.

EXAMPLE

$$\sum_{k=1}^{\infty} \frac{k}{2^k} x^k \; ; \quad a_k = \frac{k x^k}{2^k}$$

Ratio test:

$$\left| \frac{a_{k+1}}{a_k} \right| = \frac{k+1}{2k} |x|$$

$$\to \frac{|x|}{2}$$

$\Rightarrow$ The series converges for

$$-2 < x < 2 .$$

DEFINITION (The Sum Function)

$f : I \to \mathbb{R}$:

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k .$$

(I is the interval where the series converges.)