THEOREM

Let f be integrable over $[a,b]$ and continuous at $x_0 \in [a,b]$.

Then $f : [a,b] \to \mathbb{R}$,

$$F(x) = \int_a^x f(t)\,dt$$

is differentiable at x_0 and

$$F'(x_0) = f(x_0).$$

THEOREM Fundamental Theorem of Calculus

Let f be continuous over $[a,b]$ and G such that $G'(x) = f(x)$ for all $x \in [a,b]$. Then

$$\int_a^b f(x)\,dx = G(b) - G(a).$$

DEFINITE INTEGRAL

Upper and lower Riemann sums:

$f : [a,b] \longrightarrow \mathbb{R}$, $|f(x)| \leq M$ for all $x \in [a,b]$

$P = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$ (partition of $[a,b]$)

$a = x_0 < x_1 < \ldots < x_n = b$

$I_k = [x_{k-1}, x_k]$, $\Delta x_k = x_k - x_{k-1}$, $k = 1, \ldots, n$

DEFINITION

Smallest upper bound (supremum)

Greatest lower bound (infimum)

Consider: $f(x) = \frac{1}{x}$

Let $[a,b] = [1, 10]$, then

$\max f(x) = 1$

$\min f(x) = 1/10$

If $b \to \infty$, $\inf f(x) = 0$.

We define:

$G = \sup_{x \in [a,b]} f(x)$ $\quad g = \inf_{x \in [a,b]} f(x)$

$G_k = \sup_{x \in I_k} f(x)$ $\quad g_k = \inf_{x \in I_k} f(x)$

Upper sum: $\overline{S}_P = \sum_{k=1}^{n} G_k \Delta x_k$

Lower sum: $\underline{S}_P = \sum_{k=1}^{n} g_k \Delta x_k$

$g \leq g_k \leq G_k \leq G$,

$\sum_{k=1}^{n} \Delta x_k = b - a$

Thus

$$g(b-a) \le \underline{S}_p \le \overline{S}_p \le G(b-a)$$

DEFINITION

$$\sup_p \underline{S}_p = \underline{\int}_{[a,b]} f \quad \text{(lower integral)}$$

$$\inf_p \overline{S}_p = \overline{\int}_{[a,b]} f \quad \text{(upper integral)}$$

Further, if

$\underline{\int}_{[a,b]} f = \overline{\int}_{[a,b]} f$, then f is integrable and its integral is this value.

RIEMANN SUM

$$S_p = \sum_{k=1}^{n} f(\xi_k)\,\Delta x_k, \quad \xi_k \in I_k$$

Norm of p : $|p| = \max \Delta x_k$

DEFINITION

S_p has a limit A as $|p| \to 0$, if every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|p| < \delta \Rightarrow |S_p - A| < \varepsilon$$

independent of the choice of ξ_k.

$$\lim_{|p| \to 0} S_p = A.$$

THEOREM

If f is integrable, then

$$\lim_{|p| \to 0} S_p = \int_a^b f.$$

TRAPEZOIDAL RULE

$T_n[f; a, b]$; $h = \Delta x_k$

One interval :

$$\int_{x_{j-1}}^{x_j} f(x)\,dx \approx h \, \frac{y_{j-1} + y_j}{2} ,$$

where $y_j = f(x_j)$.

DEFINITION

$T_n[f; a, b] =$

$$h \left(\frac{1}{2} y_0 + \frac{1}{2} y_1 + \frac{1}{2} y_1 + \frac{1}{2} y_2 + \ldots + \frac{1}{2} y_n \right)$$

$$= \frac{h}{2} \left(y_0 + 2y_1 + \ldots + 2y_{n-1} + y_n \right)$$

EXAMPLE $I = \int_1^2 \frac{dx}{x}$; $T_4 = ?$

$$h = \frac{2-1}{4} = \frac{1}{4}$$

$$T_4 = \frac{1}{4} \left(\frac{1}{2} \cdot 1 + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2} \cdot \frac{1}{2} \right)$$

$$\simeq 0.697$$

$$P = \left\{ \underset{0}{1}, \underset{1}{\tfrac{5}{4}}, \underset{2}{\tfrac{6}{4}}, \underset{3}{\tfrac{7}{4}}, \underset{4}{2} \right\}$$

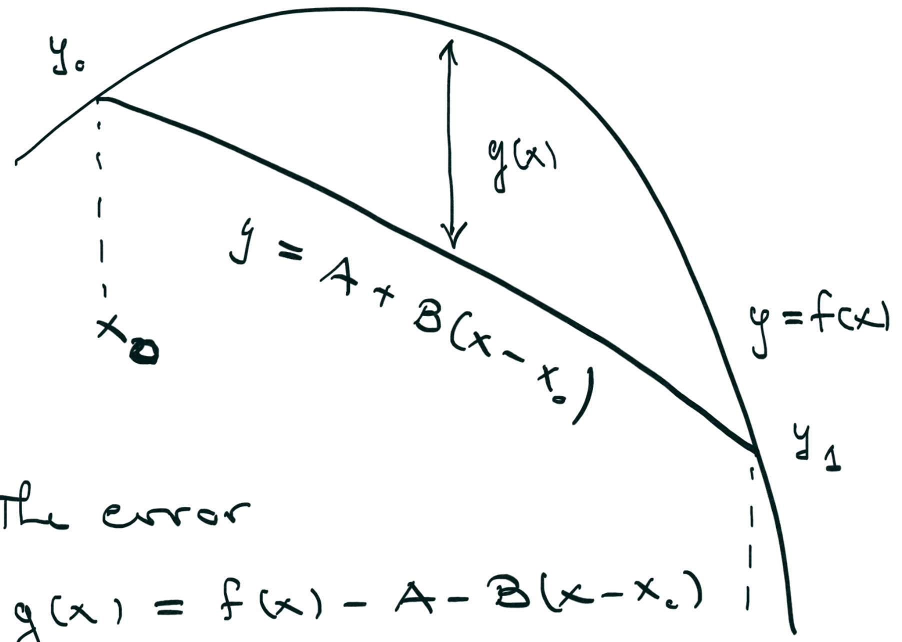
THEOREM The Error Estimate

Let f'' be continuous and bounded from above over $[a, b]$.

$|f''(x)| \leq K$ (const.) with $h = \frac{b-a}{n}$, we have

$$\left| \int_a^b f(x) - T_n \right| \leq \frac{K(b-a)^3}{12 n^2}$$

PROOF



The error

$$g(x) = f(x) - A - B(x - x_0)$$

$$= f(x) - y_0 - \frac{1}{h}(y_1 - y_0)(x - x_0)$$

$$\int g(x)\,dx = \int f(x)\,dx - h\frac{y_0 + y_1}{2}$$

We know: $g''(x) = f''(x)$

$$g(x_0) = g(x_1) = 0$$

Also:

$$\int_{x_0}^{x_1} (x - x_0)(x_1 - x)\, g''(x)\,dx = -2\int_{x_0}^{x_1} g(x)\,dx$$

Triangle inequality:

$$\left| \int_{x_0}^{x_1} f(x)\,dx - h\frac{y_0 + y_1}{2} \right| \leftarrow$$

$$\leq \frac{1}{2}\int_{x_0}^{x_1} (x - x_0)(x_1 - x)\,|f''(x)|\,dx$$

$$\leq \frac{K}{2}\int_{x_0}^{x_1} (-x^2 + (x_0 + x_1)x - x_0x_1)\,dx$$

$$= \frac{K}{12}(x_1 - x_0)^3 = \frac{K}{12}h^3$$

Summing over the whole interval:

$$\sum_{j=1}^{n} \frac{K}{12}h^3 = K \cdot \frac{1}{12} \cdot n \cdot h^3$$

$$= K\frac{(b-a)}{12}h^2 = \frac{K(b-a)^3}{12n^2} \quad \square$$