$$\int_0^a \sqrt{a^2 - x^2}\, dx = ?$$

Let $x = a \sin t$

$dx = a \cos t \, dt$

$x \in [0, a] \Rightarrow t = [0, \pi/2]$

$$\int_0^{\pi/2} a^2 \cos^2 t \, dt =$$

$$a^2 \int_0^{\pi/2} \frac{1 + \sin t \cos t}{2} = \frac{1}{4} \pi a^2$$

$$\left(a^2 \left[\frac{1 + \sin t \cos t}{2} \right]_0^{\pi/2} \right)$$

Useful identities:

f even: $\displaystyle\int_{-a}^{a} f(x)\, dx = 2 \int_0^a f(x)\, dx$

f odd: $\displaystyle\int_{-a}^{a} f(x)\, dx = 0$

f ω-periodic: $\displaystyle\int_a^b f(x)\, dx = \int_{a+\omega}^{b+\omega} f(x)\, dx$

INTEGRATION TECHNIQUE : METHOD OF SUBSTITUTION

Definite integrals :

↳ Definition (Riemann sum)

↳ Numerical methods

↳ Trapezoidal

Midpoint

. . .

Fundamental Theorem of Calculus :

↳ "anti-derivative"

Techniques :

(1) substitution

(2) integration by parts

Indefinite integral :

$$F'(x) = f(x)$$

$$F(x) = \int_a^x f(t)\,dt$$

over closed interval $[a, x]$; f continuous.

We know that the derivative of a constant function is zero.

$$\int f(x)\,dx = F(x) + C,$$

C constant.

METHOD OF SUBSTITUTION

$$\int f(x)\,dx = \int F'(x)\,dx$$

$$= F(x) + C$$

Set $x = g(t)$

$$= F(g(t)) + C$$

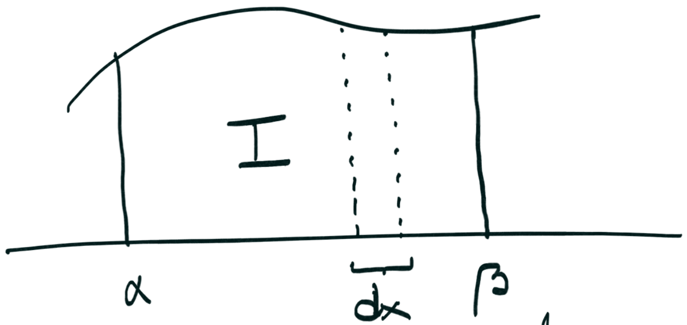
$$= \int \frac{d}{dt} F(g(t))\, dt$$

$$= \int F'(g(t))\, g'(t)\, dt$$

$$= \int f(g(t))\, \underbrace{g'(t)\, dt}$$

What is going on ?

Definite integral :



$\alpha = g(t_1)$

$\beta = g(t_2)$ $\Rightarrow \displaystyle\int_{t_1}^{t_2} \dots\, dt$

Notice : $dx = g'(t)\, dt$

EXAMPLE

$$\int \frac{dx}{x^2+a^2}, \quad a > 0$$

Let $x = at$; $dx = a\, dt$

$$\int \frac{dx}{x^2+a^2} = \int \frac{a\, dt}{a^2t^2+a^2} = \int \frac{\cancel{a}\, dt}{a^{\cancel{2}}(t^2+1)}$$

$$= \frac{1}{a} \int \frac{dt}{t^2+1} = \frac{1}{a}\, \overline{\arctan}\, t + C$$

$$= \frac{1}{a}\, \overline{\arctan} \frac{x}{a} + C$$

What about :

$$\int_{\alpha}^{\beta} \frac{dx}{x^2+a^2} = \frac{1}{a} \int_{\alpha/a}^{\beta/a} \frac{dt}{t^2+1}$$

HYPERBOLIC FUNCTIONS

DEFINITION

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$D \cosh x = \frac{e^x - e^{-x}}{2} = \sinh x$$

$$D \sinh x = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

Inverse functions :

EXAMPLE

$$\operatorname{arsinh} x = y \iff x = \sinh y$$

$$\underline{\underline{x}} = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$= \underline{\underline{\frac{(e^y)^2 - 1}{2e^y}}}$$

We get : $\leftarrow$

$$(e^y)^2 - 2x\,e^y - 1 = 0$$

$$\Rightarrow e^y = x \pm \underbrace{\sqrt{x^2+1}}_{\geq 1}$$

Since $e^y > 0$, the only choice is

$$e^y = x + \sqrt{x^2+1}$$

$$\Rightarrow y = \ln\left(x + \sqrt{x^2+1}\right)$$
$$= \operatorname{arsinh} x$$

MORE EXAMPLES:

$$\int \frac{dx}{x^2-a^2}, \quad a \neq 0$$

Let $x = at$; $dx = a\,dt$

$$= \frac{1}{a} \int \frac{dt}{t^2-1} = I$$

Consider:

$$\frac{1}{t^2-1} = \frac{1}{(t+1)(t-1)}$$

$$= \frac{A}{t+1} + \frac{B}{t-1}$$

$$1 = A(t-1) + B(t+1)$$

t^0: $1 = -A + B$

t^1: $0 = A + B$

$$\Rightarrow A = -\frac{1}{2}, \quad B = \frac{1}{2}$$

$$I = \frac{1}{2a} \int \left(\frac{1}{t-1} - \frac{1}{t+1} \right) dt$$

$$= \frac{1}{2a} \left(\ln|t-1| - \ln|t+1| \right) + C$$

$$= \frac{1}{2a} \ln\left| \frac{t-1}{t+1} \right| + C$$

$$= \frac{1}{2a} \ln\left| \frac{x-a}{x+a} \right| + C$$

$$\int \frac{\overbrace{x\,dx}}{x^4+1} = ?$$ Let $x^2 = t$; $\underbrace{2x\,dx} = dt$

$$= \frac{1}{2} \int \frac{dt}{t^2+1} = \frac{1}{2} \arctan t + C$$

$$= \frac{1}{2} \arctan x^2 + C$$