INTEGRATION BY PARTS

Product rule:

$$\frac{d}{dx}\left[u(x)v(x)\right] = u'(x)v(x) + u(x)v'(x)$$

By integrating (and rearranging):

$$\int u'(x)v(x)\,dx = u(x)v(x) - \int u(x)v'(x)\,dx$$

EXAMPLE $\int x^2 e^x\,dx = I$

$$\int \underbrace{x^2}_{v}\,\underbrace{e^x}_{u'}\,dx = \underbrace{x^2}_{v}\,\underbrace{e^x}_{u} - \int \underbrace{2x}_{v'}\,\underbrace{e^x}_{u}\,dx$$

$$= I$$

⊛ $2\int \underbrace{x}_{v}\underbrace{e^x}_{u'}\,dx = 2xe^x - 2\int e^x\,dx$

Combining:

$$I = x^2e^x - 2xe^x + 2e^x + C$$

$$= e^x(x^2 - 2x + 2) + C$$

EXAMPLE

$$\int \ln x\,dx = ?$$

$$I = \int \underbrace{1}_{u'} \cdot \underbrace{\ln x}_{v}\,dx$$

$$= x\ln x - \int \underbrace{x \cdot \frac{1}{x}}_{=1}\,dx$$

$$= x\ln x - x + C$$

$D(x\ln x - x + C) =$

$\ln x + x\cdot\cancel{\frac{1}{x}} - \cancel{1} = \ln x$

THEOREM

Every rational function can be integrated in closed form.

Rational function:

$\frac{H(x)}{Q(x)}$, where $H(x)$ and $Q(x)$ are polynomials.

EXAMPLE

$$\int \frac{dx}{x(x^6+1)^2} = I$$

Substitution: $x^6 = t$; $6x^5 dx = dt$

$$I = \frac{1}{6}\int \frac{6x^5 dx}{x^6(x^6+1)^2} = \frac{1}{6}\int \frac{dt}{t(t+1)^2}$$

PARTIAL FRACTION DECOMPOSITION:

$$\frac{1}{t(t+1)^2} = \underline{\underline{\frac{A}{t}}} + \underline{\underline{\frac{B}{(t+1)^2}}} + \underline{\underline{\frac{C}{(t+1)}}}$$

$$= \frac{A(t+1)^2 + Bt + Ct(t+1)}{t(t+1)^2}$$

Note: 2^{nd} order polynomial:

$$ax^2 + bx + c = p(x)$$

has a basis $\{x^0, x^1, x^2\}$.

Now we have:

$$1 = A(t+1)^2 + Bt + Ct(t+1)$$

t^2: $0 = A + C$

t^1: $0 = 2A + B + C$

t^0: $1 = A$

$A = 1$, $C = -1$, $B = -1$

Integrate:

$$I = \frac{1}{6}\left(\ln|t| + \frac{1}{t+1} - \ln|t+1|\right) + C$$

$$= \frac{1}{6}\left(\ln\frac{x^6}{x^6+1} + \frac{1}{x^6+1}\right) + C$$

EXAMPLE

$$\int \frac{x^4+1}{x^3-x^2+x-1}\,dx = (\text{long division})$$

$$\int\left(x+1+\frac{2}{x^3-x^2+x-1}\right)dx$$

$$= \frac{1}{2}x^2 + x + \int \frac{2}{(x-1)(x^2+1)}\,dx$$

$$\frac{2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}$$

$$2 = A(x^2+1) + (Bx+C)(x-1)$$

x^2: $0 = A + B$

x^1: $0 = \quad -B + C$

x^0: $2 = A \quad - C$

$A = 1,\ B = -1,\ C = -1$

$$I = \frac{1}{2}x^2 + x + \int \frac{dx}{x-1} - \int \frac{x+1}{x^2+1}\,dx$$

$$= \frac{1}{2}x^2 + x + \int \frac{dx}{x-1} - \int \frac{x\,dx}{x^2+1} - \int \frac{dx}{x^2+1}$$

$$= \frac{1}{2}x^2 + x + \ln|x-1| - \frac{1}{2}\ln(x^2+1) - \arctan x + C$$

EXAMPLE $\quad (n \in \mathbb{N})$

$$I_n = \int x^n e^x \, dx = x^n e^x - n \int x^{n-1} e^x \, dx$$

$$= x^n e^x - n I_{n-1}$$

Recursion!

This terminates, since

$$I_0 = \int e^x \, dx = e^x + C$$

EXAMPLE

$$I = \int e^{kx} \sin(nx) \, dx, \quad (k \neq 0)$$

$$= \frac{1}{k} e^{kx} \sin(nx) - \int \frac{n}{k} e^{kx} \cos(nx) dx \quad (1)$$

$$= \frac{1}{k} e^{kx} \sin(nx) - \frac{n}{k} \left[\frac{1}{k} e^{kx} \cos(nx) - \int \frac{n}{k} e^{kx} \sin(nx) \, dx \right] \quad (2)$$

$$= \frac{1}{k} e^{kx} \sin(nx) - \frac{n}{k^2} e^{kx} \cos(nx) - \frac{n^2}{k^2} I \quad (3)$$

$$\Rightarrow \left(1 + \frac{n^2}{k}\right) I = \frac{1}{k} e^{kx} \sin(nx) - \frac{n}{k^2} e^{kx} \cos(nx)$$

$$I = \frac{k \sin(nx) - n \cos(nx)}{k^2 + n^2} e^{kx} + C$$