Improper Integrals :

Two cases :

(a) the interval is infinite

(b) the function is not bounded over the whole interval

EXAMPLE

$$\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \lim_{\substack{a \to -\infty \\ b \to \infty}} \int_a^b \frac{dx}{x^2+1}$$

$$= \lim_{\substack{a \to -\infty \\ b \to \infty}} \Big/_a^b \overline{\arctan}\, x =$$

$$= \lim_{b \to \infty} \overline{\arctan}\, b - \lim_{a \to -\infty} \overline{\arctan}\, a$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

(b) similarly at the jumps

ODEs

Ordinary differential equations :

For example :

Newton's Law : $F = ma$

$$\iff F = m \frac{d^2 s}{dt^2}$$

(a = acceleration , s = displacement)

This is an example of a 2^{nd} order ODE.

General 1^{st} order ODE :

$$\frac{d\, y(x)}{dx} = f(x, y(x))$$

The solution curve is $y(x)$.

At every point $(x, y(x))$ the slope of the solution curve is $f(x, y(x))$.

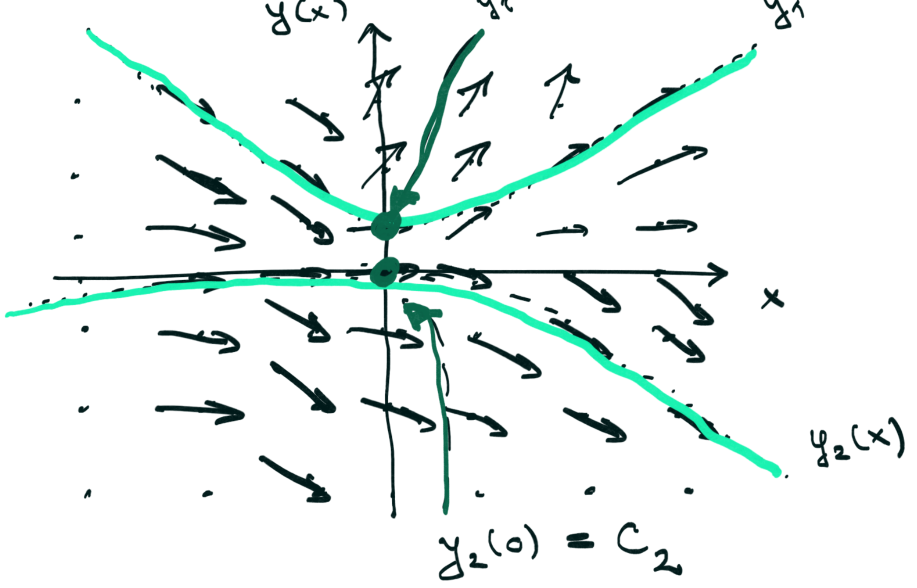
Equation: $y' = f(x, y(x))$

Observation: We only have information up to a constant!

Terminology:

- the general solution includes all possible solutions
- the initial conditions lead to particular solutions

PHASE PORTRAIT



SEPARABLE EQUATIONS:

$$\frac{dy}{dx} = f(x)\,g(y)$$

Formal equation:

$$\frac{dy}{g(y)} = f(x)\,dx$$

By integrating:

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C$$

EXAMPLE $\quad \frac{dy}{dx} = \frac{x}{y}$

Here: $f(x) = x$, $g(y) = \frac{1}{y}$

Thus,

$$\int y\,dy = \int x\,dx + \tilde{C}$$

$$\Rightarrow \frac{1}{2}y^2 = \frac{1}{2}x^2 + \tilde{C}$$

$$\Longleftrightarrow \underline{y^2 - x^2} = 2\tilde{C} = \underline{C}$$

The solution curves are hyperbolae, with asymptotes: $y = x$, $y = -x$, corresponding the case $C = 0$.

$$\left(y^2 - x^2 = (y-x)(y+x) = 0 \right)$$

LINEAR 1ST ORDER ODE:

$$\frac{dy}{dx} + p(x)\, y = q(x)$$

If $q(x) = 0$, homogeneous,

$q(x) \neq 0$, non homogeneous.

$\frac{dy}{dx} + p(x)\, y = 0$ is separable:

$$y = K e^{-\mu(x)}, \quad \mu(x) = \int p(x)\, dx$$

$$\frac{d\mu(x)}{dx} = p(x)$$

Notice: $f(x) = p(x)$

$g(y) = -y$

Formally: $L = \frac{d\cdot}{dx} + p(x)\,\cdot$

Previously: $Df(x) = f'(x)$

Here: $L(y) = q(x)$

Let y_h be the solution of the homogeneous solution, i.e.,

$$L(y_h) = 0.$$

Let y_p be a particular solution of the ODE, i.e.,

$$L(y_p) = q(x).$$

Now:

$$L(y_p) + L(y_h) = q(x)$$

or

$$L(y_p + y_h) = q(x)$$

$$\frac{d(y_p + y_h)}{dx} + p(x)(y_p + y_h) =$$

$$= \frac{dy_p}{dx} + \underline{\underline{\frac{dy_h}{dx}}} + p(x)\,y_p + \underline{\underline{p(x)\,y_h}}$$

$$= q(x)$$

Solution methods :

(A) Integrating factor

$$\frac{d}{dx}\left(e^{\mu(x)}\,y(x)\right) =$$

$$e^{\mu(x)}\frac{dy}{dx} + e^{\mu(x)}\frac{d\mu}{dx}\,y(x)$$

$$= e^{\mu(x)}\left(\frac{dy}{dx} + p(x)\,y(x)\right)$$

$$= e^{\mu(x)}\,q(x)$$

Integrate!

We get :

$$e^{\mu(x)}\,y(x) = \int e^{\mu(x)}\,q(x)\,dx$$

$$\Rightarrow \underline{y(x) = e^{-\mu(x)} \int e^{\mu(x)}\,q(x)\,dx}$$