DEFINITION EULER'S METHOD (Implicit)

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + h f(x_{n+1}, y_{n+1})$$

NOTE: Every step requires solving an equation.

2^{ND} ORDER ODEs :

$$\phi(x, y, y', y'') = 0 \quad \text{(implicit)}$$

$$y'' = f(x, y, y') \quad \text{(explicit)}$$

Solution: $y = \varphi(x, c_1, c_2)$

Initial value problem:

$$y(x_0) = y_0 \ ; \ y'(x_0) = P_0$$

Boundary value problem:

$$y(x_1) = y_1 \ ; \ y(x_2) = y_2$$

THEOREM

$y'' = f(x, y, y')$ and

$$\begin{cases} y' = z \\ z' = f(x, y, z) \end{cases}$$

are equivalent.

ODEs

$$\frac{dy}{dx} + p(x)y = q(x)$$

$q(x) = 0$ (homog.)

$q(x) \neq 0$ (non homog.)

Respective solutions: y_h, y_p

$$L(y_h + y_p) = q(x)$$

$$L = \frac{d\cdot}{dx} + p(x)\cdot$$

$$\frac{dy}{dx} + p(x)y = 0$$

$$\iff \frac{dy}{dx} = -p(x)y$$

$$\Rightarrow \int \frac{dy}{y} = -\int p(x)dx$$

$$\Rightarrow \ln y = -\mu(x) + C$$

$$\Rightarrow y = Ke^{-\mu(x)}$$

Notation: $\mu(x) = \int p(x)dx$

$$\frac{d\mu(x)}{dx} = p(x)$$

Integrating factor:

$$\frac{d}{dx}\left(\underline{\underline{e^{\mu(x)}}}\,y(x)\right) =$$

$$= e^{\mu(x)}\left(\frac{dy}{dx} + p(x)y\right) = e^{\mu(x)}q(x)$$

Hence, $e^{\mu(x)} y(x) = \int e^{\mu(x)} q(x)\,dx$

EXAMPLE

$$\frac{dy}{dx} + \frac{y}{x} = 1 \,, \quad x > 0$$

$p(x) = \frac{1}{x}$; $\mu(x) = \int \frac{dx}{x} = \ln x$

Therefore:

$$e^{\mu(x)} = x$$

So,

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y$$

$$= x\left(\frac{dy}{dx} + \frac{y}{x}\right) = x \cdot 1$$

$$= x$$

$$\Rightarrow \quad xy = \frac{1}{2}x^2 + c$$

$$\Rightarrow \quad y = \frac{x}{2} + \frac{c}{x}$$

(B) Variation of the parameter:

$$K = K(x)$$

$$\frac{d}{dx}\left(K(x)e^{-\mu(x)}\right) + p(x)K(x)e^{-\mu(x)} = q(x)$$

$$\Rightarrow K'(x)e^{-\mu(x)} - \cancel{K(x)\mu(x)}e^{-\mu(x)} + \cancel{p(x)K(x)}e^{-\mu(x)} = q(x)$$

?

$$\Rightarrow K'(x) = e^{\mu(x)} q(x)$$

Returning to our EXAMPLE:

$$y = Ke^{-\mu(x)} = \frac{K}{x}$$

We get: $K'(x) = x \cdot 1 = x$

$$K(x) = \frac{1}{2}x^2 + c$$

Solution: $\frac{1}{x}\left(\frac{1}{2}x^2 + c\right) =$

$$y = \frac{x}{2} + \frac{c}{x} \qquad (!)$$

Comment on linearity:

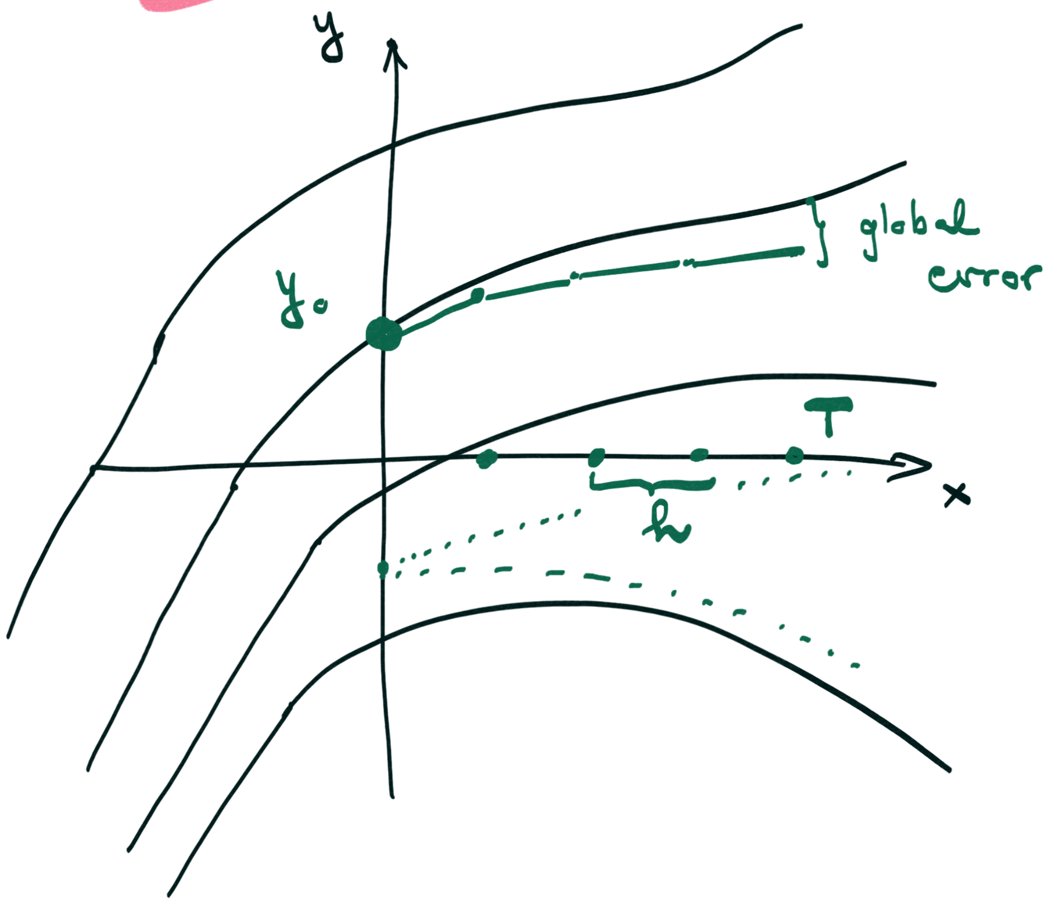
Consider normal differentiation:

$$\begin{cases} D(x^2 + x) = D(x^2) + D(x) \\ \qquad\qquad\;\; = 2x + 1 \\ D(2x) = 2D(x) = 2 \end{cases}$$

Here, L is linear.

$$\begin{tikzcd} y_p(x) \arrow[rr, "L"] & & g(x) \\ + & & + \\ y_h(x) \arrow[rr, dotted] & & 0 \end{tikzcd}$$

NUMERICAL SOLUTION OF ODEs



$$\begin{cases} \dfrac{dy}{dx} = f(x,y) \\ y(x_0) = y_0 \end{cases}$$

DEFINITION EULER'S METHOD (explicit)

$$x_{n+1} = x_n + h$$

$$y_{n+1} = y_n + h f(x_n, y_n)$$

EXAMPLE

$$\frac{dy}{dx} = x - y \quad , \quad y(0) = 1$$

Interval: $[0, 1]$,

$h = 1/5$,

$$y_{n+1} = y_n + \frac{1}{5}(x_n - y_n)$$

$$= \frac{4}{5} y_n + \frac{1}{5} x_n$$

$$\left(y(x) = x - 1 + 2e^{-x} \right)$$

At $x_n = 1$:

Error $e_n = y(x_n) - y_n$

~ 0.08

DEFINITION MODIFIED EULER'S METHOD (HEUN'S METHOD)

$$x_{n+1} = x_n + h$$

$$u_{n+1} = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + h\left(\frac{1}{2}\left(f(x_n, y_n) + f(x_{n+1}, u_{n+1})\right)\right)$$

"Predictor - Corrector" - method

$$\downarrow \qquad\qquad \downarrow$$

$$u_{n+1} \qquad\qquad y_{n+1}$$