2nd ORDER ODE WITH CONSTANT COEFFICIENTS

Consider $y'' + ay' + by = 0$.

a, b are scalars, $a, b \in \mathbb{R}$.

Let us try $y = e^{rx}$!

We have:

$$y = e^{rx}, \quad y' = re^{rx}, \quad y'' = r^2 e^{rx}$$

Substituting to the original equation:

$$r^2 e^{rx} + are^{rx} + be^{rx} = 0$$

$$\Longleftrightarrow \quad r^2 + ar + b = 0$$

We have derived an auxiliary equation!

Hence,

$$r = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$$

Three different cases:

(i) $a^2 - 4b > 0$:

Two distinct roots r_1, r_2

(ii) $a^2 - 4b = 0$:

Double root $r_{1,2} = -\frac{a}{2}$

(iii) $a^2 - 4b < 0$:

Complex conjugate pair

$$r_{1,2} = \alpha \pm i\beta$$

(i is the imaginary unit)

Homogeneous problem :

corresponding solutions

(i) $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$

(ii) $y(x) = (c_1 + c_2 x) e^{-\frac{a}{2}x}$

(iii) $y(x) = e^{\alpha x}(c_1 \sin\beta x + c_2 \cos\beta x)$

Observation :

The equation $y'' + ay' + by = R(x)$ can always be solved with applications of two quadrature rules.

Common types of RHSs :

1) $R(x)$ = polynomial of degree n

Method of undetermined coefficients

$\longrightarrow$ just like with the partial fractions

2) $R(x) = A e^{\lambda x}$

Try : $y_0(x) = K e^{\lambda x}$

Get : $y_0(x) = \dfrac{A}{\lambda^2 + a\lambda + b} e^{\lambda x}$

Possible problem :

What if λ is one of the roots of the auxiliary equation ?

Try again with

$y_0(x) = K x^m e^{\lambda x}$ where m is the order of the root.

EXAMPLE

$$y'' + 2y' + y = e^{-x}$$

Homog. $y_H = (c_1 + c_2 x) e^{-x}$

Particular: char: $(1+\lambda)^2 = 0$

$\Rightarrow -1$ is a double root

Try: $y_0(x) = K x^2 e^{-x}$

$$y_0'(x) = K(2x - x^2)e^{-x}$$

$$y_0''(x) = K(2 - 4x + x^2)e^{-x}$$

Substitute to the equation and solve for K:

Here $K = 1/2$

The general solution:

$$y = y_H + y_0 = \left(c_1 + c_2 x + \tfrac{1}{2}x^2\right)e^{-x}$$

3) $R(x) = A \sin \omega x + B \cos \omega x, \quad \omega \neq 0$

Try: $y_0(x) = K \sin \omega x + L \cos \omega x$

Special case: $a = 0, \; b = \omega^2$

We get: $y'' + \omega^2 y = A \sin \omega x + B \cos \omega x$

$\Rightarrow$ RESONANCE (!)

Try:

$$y_0(x) = Kx \sin \omega x + Lx \cos \omega x$$

EXAMPLE

$$y'' + 4y = \sin 2t$$

$$y_0(t) = Kt \sin 2t + Lt \cos 2t$$

$$y_0'(t) = (K - 2Lt)\sin(2t) + (L + 2Kt)\cos(2t)$$

$$y_0''(t) = -4(L + Kt)\sin(2t) + 4(K - Lt)\cos(2t)$$

Substituting:

$$y_0'' + 4y_0 = \sin 2t$$

$$-4(L+Kt)\sin(2t) + 4(K-Lt)\cos 2t$$
$$+ 4Kt\sin(2t) + 4Lt\cos(2t)$$
$$= \sin(2t)$$

$\sin 2t$: $\quad -4L - 4Kt + 4Kt = 1$

$\cos 2t$: $\quad -4Lt + 4K + 4Lt = 0$

?

$$\Rightarrow L = -\frac{1}{4}$$

We should get $K = 0$

General solution:

$$y = y_H + y_0 =$$

$$c_1 \cos 2t + c_2 \sin 2t - \frac{1}{4} t \cos 2t$$