5th Lecture: Elements of Classical Reliability Theory

Aim of the present lecture

The aim of the present lecture is to introduce the basic elements of the classical *reliability theory*. First the problem of assessing the reliability of components and systems, based on observed times till failure, is addressed and the important concept of failure rates is introduced. Thereafter it is illustrated how such failure rates may be updated in a Bayesian framework based on additional information. Subsequently some generic data on failure rates are provided for electrical and mechanical components and systems. Finally an introduction is given to the structural reliability theory. This theory is especially applicable for the reliability analysis of components and systems, such as e.g. building structures, for which in general it is not possible to achieve relevant information on the time till failure. A more elaborate treatment of the methods of structural reliability is provided in a separate lecture. On the basis of the present lecture, it is expected that the students should acquire knowledge and skills in regard to:

- For which types of components and systems can the *reliability* be assessed on the basis of observed failure data?
- What is a reliability function?
- What is a *failure rate* function?
- How can the failure rate be estimated based on observed times till failure?
- How can failure rates be updated based on additional information?
- What is a *hazard function*?
- When is it relevant to use methods of structural reliability?
- What is understood by the fundamental case?
- What is a *safety margin*?
- What is the interpretation of the *reliability index*?

5.1 Introduction

Reliability analysis of technical components and systems became a central issue during the Second World War where significant problems were encountered especially in regard to the performance of electrical systems. As an example the war systems of the, at that time modern battle ships, were reported non-operable in up to about 40 % of the time. This situation which could be quite critical in times of war was caused predominantly by failures of electrical components (radio bulbs, etc.) and the efforts initiated at that time in order to improve the performance of the electrical systems may be seen as an initiation point for the analysis of the reliability of technical components.

Since then reliability analysis of technical components and systems has been further developed and adapted for application in a wide range of different industries including the aeronautical industry, the nuclear industry, the chemical industry, the building industry and the process industry. It is important to appreciate that reliability analysis is only one of the constituents of a *decision analysis* or more popularly speaking *risk assessment*, namely the part which is concerned about the quantification of the probability that a considered component or system is in a state associated with adverse consequences, e.g. a state of failure, a state of damage or partial function, etc. The theoretical basis for reliability analysis is thus the theory of probability and statistics and derived disciplines such as operations research, systems engineering and quality control.

Classical reliability theory was, as previously indicated, developed for systems consisting of a large number of components of the same type under the same loading and which for all practical matters behaved statistically independent. The probability of failure of such components and systems can be interpreted in terms of failure frequencies observed from operation experience. Furthermore, due to the fact that failure of the considered type of components develops as a direct consequence of an accumulating deterioration process the main focus was directed towards the formulation of probabilistic models for the estimation of the statistical characteristics of the time until component failure. Having formulated these models the observed relative failure frequencies can be applied as basis for their calibration.

In structural reliability analysis the situation is fundamentally different due to the fact that structural failures are very rare and tend to occur as a consequence of an extreme event such as e.g. an extreme loading exceeding the load carrying capacity i.e. the resistance, which possibly is reduced due to deterioration such as e.g. corrosion or fatigue. In addition to this no useful information can be collected in regard to relative failure frequencies as almost all structural components and systems are unique either due to differences in the choice of material and geometry or by differences in the loading and exposure characteristics. When considering the estimation of failure probabilities for structural components it is thus necessary to establish a probabilistic modelling of both the resistances and the loads and to estimate the probability of failure on the basis of these. In this process due account must be given to the inclusion of all available statistical information concerning the material properties and the load characteristics.

In the following sections an introduction shall first be given of the classical reliability theory and thereafter consider the problem of structural reliability analysis with a view to the special characteristics of this problem.

5.2 Introduction to the classical reliability theory

Classical reliability analysis was developed to estimate the statistical characteristics of the lives of technical systems and components. These characteristics include the expected failure rate, the expected life and the mean time between failures.

Modelling the considered system by means of logical trees where the individual components are represented by the nodes it is possible to assess the key characteristics regarding the system performance including e.g. the probability that a system will fail during a specified period, the positive effect of introducing redundancy into the system and the effect of inspections and maintenance activities.

The probability of failure of a component is expressed by means of the reliability function $R_T(t)$ defined by:

$$R_T(t) = 1 - F_T(t) = 1 - P(T \le t) \tag{5.1}$$

where T is a random variable describing the time till failure and $F_T(t)$ is its cumulative distribution function. If the probability density function for T, i.e. $f_T(t)$, is known the reliability function may be defined alternatively by:

$$R_T(t) = 1 - \int_0^t f_T(\tau)d\tau = \int_t^\infty f_T(\tau)d\tau \tag{5.2}$$

The reliability function thus depends on the type of the probability distribution function for the time till failure. In the same way as when considering the probabilistic modelling of load and resistance variables, prior information may be utilised when selecting the distribution type for the modelling of the random time till failure for a technical component. The appropriate choice of distribution function then depends on the physical characteristics of the deterioration process causing the failure of the component.

In the literature several models for the time till failure have been derived on the basis of the characteristics of different deterioration processes. These include the exponential distribution, the Weibull distribution, and the Birnbaum and Saunders distribution. In case of a Weibull distribution the reliability function has the following form:

$$R_T(t) = 1 - F_T(t) = 1 - (1 - \exp\left[-(\frac{t}{k})^\beta\right]) = \exp\left[-(\frac{t}{k})^\beta\right], \qquad t \ge 0 \tag{5.3}$$

Having defined the reliability function $R_T(t)$ the expected life may be derived as:

$$E[T] = \int_0^\infty \tau \cdot f_T(\tau)d\tau = \int_0^\infty R_T(t)dt \tag{5.4}$$

which may be seen by performing the integrations in parts, provided that $\lim_{t\to\infty} t \cdot R_T(t) = 0$:

$$\begin{aligned} E[T] &= \int_0^\infty \tau f_T(\tau) dt = \left[t \cdot F_T(t)\right]_0^\infty - \int_0^\infty F_T(\tau) d\tau \\ &= \left[t \cdot (1 - R_T(t))\right]_0^\infty - \int_0^\infty (1 - R_T(\tau)) d\tau \\ &= \left[t\right]_0^\infty - \left[t \cdot R_T(t)\right]_0^\infty - \left[t\right]_0^\infty + \int_0^\infty R_T(\tau) d\tau \\ &= \int_0^\infty R_T(\tau) d\tau - \left[t \cdot R_T(t)\right]_0^\infty = \int_0^\infty R_T(\tau) d\tau \end{aligned} \tag{5.5}$$

The failure rate is a measure of how the probability of failure changes as a function of time. The failure rate thus depends on the reliability function $R_T(t)$. The probability of failure within any given interval $\left[t, t + \delta \text{ t}\right]$ is the probability that the actual life lies in the interval and is thus given as:

$$P(t < T \leq t + \delta\ t) = F_T(t + \delta \text{ t}) - F_T(t) = R_T(t) - R_T(t + \delta \text{ t}) \tag{5.6}$$

The failure rate function $z(t)$ being the average rate at which failures occur in a given time interval provided that the considered component has not failed prior to the interval is thus:

$$z(t) = \frac{R_T(t) - R_T(t + \delta \text{ t})}{\delta \ \ t \, R_T(t)} \tag{5.7}$$

The failure rate function for most technical systems is known as the *bath-tub curve* illustrated in Figure 5.1.

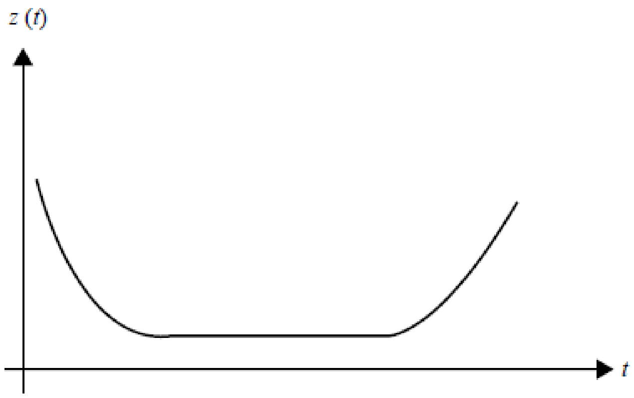

Figure 5.1: Illustration of a failure rate function – the bath-tub curve.

The bath-tub curve is typical for many technical components where in the initial phase of the life the birth defects, production errors etc. are a significant source of failure. When the component has survived a certain time it implies that birth defects are not present and consequently the reliability increases. Thereafter a phase of steady state is entered and subsequently a phase of ageing. The steepness of the ageing part of the failure rate function is important. The more pronounced and the steeper the transition is from the steady phase to the ageing phase of the life of the component the more obvious is the decision on when to exchange or maintain the component.

The shape of the failure rate function has also implications on the meaningful inspection strategies, which may be implemented as a means for condition control of a component. For components exhibiting a constant failure rate function, i.e. components with an exponential distribution as given in Equation (5.8) for the time till failure, inspections are of little use.

$$f_T(t) = z \cdot \exp(-z \cdot t)) \tag{5.8}$$

In this case the component does not exhibit any degradation and there is not really anything to inspect. However, for components with a slowly increasing failure rate function inspections may be useful and can be planned such that the failure rate does not exceed a certain critical level. If the failure rate function is at first quasi constant and then followed by an abrupt increase, inspections are also of little use. However, in this case, a replacement strategy may be more appropriate.

The hazard function $h(t)$ is defined through the instantaneous failure rate as the considered interval approaches zero. Thus the hazard function is given as:

$$h(t) = \lim_{\delta\ t \to 0} \frac{R_T(t) - R_T(t + \delta\ t)}{\delta\ \ t\ R_T(t)} = \frac{1}{R_T(t)}\left[-\frac{d}{dt}R_T(t)\right] = \frac{f_T(t)}{R_T(t)} \tag{5.9}$$

and the probability that a component having survived up till the time t will fail in the next small interval of time dt is then $h(t)dt$.

An important issue is the assessment of failure rates on the basis of observations. As mentioned previously data on observed failure rates may be obtained from databanks of failures from different application areas. Failure rates may be assessed on the basis of such data by:

$$z = \frac{n_f}{\tau \cdot n_i} \tag{5.10}$$

where n_f is the number of observed failure in the time interval τ and n_i is the number of components at the start of the considered time interval. Care must be exercised when evaluating failure rates on this basis. If the components are not new in the beginning of the considered time interval the failure rates may be overestimated and if the interval is too short no observed failures may be present. For such cases different approaches to circumvent this problem may be found in the literature, see e.g. Stewart and Melchers (1997). Alternatively the failure rates may also be assessed by means of e.g. Maximum-Likelihood estimation where the parameters of the selected probability distribution function for the time till failure are estimated on the basis of observed times till failures.

Due to the lack of data and general uncertainties associated with the applicability of the available data for a specific considered case, failure rates may themselves be modelled as uncertain. The basis for the a-priori assessment of the uncertainty associated with the failure rates may be established subjectively or preferably as a bi-product of the Maximum-Likelihood estimation of the distribution parameters of the probability distribution function for the time till failure. Having established an a-priori model for the failure rate for a considered type of component another important issue is how to update this estimate when new or more relevant information about observed failures become available.

Applying the rule of Bayes the posterior probability density function for the failure rate may be established as:

$$f_Z''(z|\mathbf{t}) = \frac{L(\mathbf{t}|z)\cdot f_Z'(z)}{\int_0^{\infty} L(\mathbf{t}|z)\cdot f_Z'(z)\,dz} \tag{5.11}$$

Assuming that the time till failure for a considered component is exponential distributed the likelihood function is given as:

$$L(\mathbf{t}|z) = \prod_{i=1}^{n} z\exp(-z\cdot t_i) \tag{5.12}$$

Example 5.1 – Pump failure modelling

For the purpose of illustration a risk analysis of an engineering system including a number of pumps is being performed. As a basis for the estimation of the probability of failure of the individual pumps in the system, frequentistic data on pump failures are analysed. From the manufacturer of the pumps it is informed that a test has been made where 10 pumps were put in continuous operation until failure. The results of the tests are given in Table 5.1, where the times till failure (in years) for the individual pumps are given.

Pump	Time till failure
1	0.24
2	3.65
3	1.25
4	0.2
5	1.79
6	0.6
7	0.74
8	1.43
9	0.53
10	0.13

Table 5.1: Observed time till failure for a considered type of pumps.

Based on the data in Table 5.1 the annual failure rate for the pumps must be estimated with and without using the assumption that the times between failure is exponentially distributed.

Based on the data alone the sample mean value of the observed times till failure is calculated. This yields 1.06 years and the number of failures per year (failure rate) z is thus the reciprocal value equal to 0.95.

If it is assumed that only data from pumps failed within the first year are available the corresponding failure rate is 2.46. If it is assumed that the times till failure are exponentially distributed the Maximum Likelihood Method can be used to estimate the failure rate.

The probability density function for the time till failure may be written as:

$$f_T(t) = z\exp(-z \cdot t) \tag{5.13}$$

The log-Likelihood is written as

$$l(\mathbf{t}|z) = \sum_{i=1}^{10} \left(\ln(z) - z \cdot t_i\right) \tag{5.14}$$

where t_i are the observed times till failure.

By maximising the log-Likelihood function with respect to z using all observations in Table 5.1 a failure rate equal to 0.95 is obtained, which is identical to the rate found above using all observations. If only the observations where failure occur within the first year are used in the Maximum Likelihood estimation, a failure rate equal to 2.45 is obtained, close to the value obtained above using only the data from the first year.

It thus seems that if only the observations of failure from the first year are available – which indeed could be the situation in practice – the failure rate is estimated rather imprecisely. However there is one approach, still using the Maximum Likelihood method, whereby this problem can be circumvented to a large degree. If the log-Likelihood function is formulated as:

$$l(\mathbf{t}|z) = n_n \ln(1 - F_T(1)) + \sum_{i=1}^{n_f} \ln(z) - z \ \ t_i = -n_n z \ + \sum_{i=1}^{n_f} \ln(z) - z \ \ t_i \tag{5.15}$$

where n_n is the number of pumps not failed within the first year and n_f is the number of pumps failed within the first year, and furthermore the probability distribution function of the time till failure in the first year is given as:

$$F_T(t) = 1 - \exp(-z \ \ t) \quad => \quad F_T(1) = 1 - \exp(-z) \tag{5.16}$$

An estimate of the failure rate equal to 0.93 is then obtained which is significantly better than when not utilising the information that a number of the pumps did not experience failure within the first year.

Using the Maximum Likelihood Method has the advantage that the uncertainty associated with the estimated parameters is readily provided through the second order partial derivative of the log-Likelihood function. Furthermore the estimated parameters may be assumed Normal distributed.

Using all samples in the estimation the uncertain failure rate may then be found to be Normal distributed with mean value equal to 0.95 and standard deviation equal to 0.42. The (prior) probability density function for the uncertain failure rate $f_Z^{'}(z)$ is illustrated in Figure 5.2.

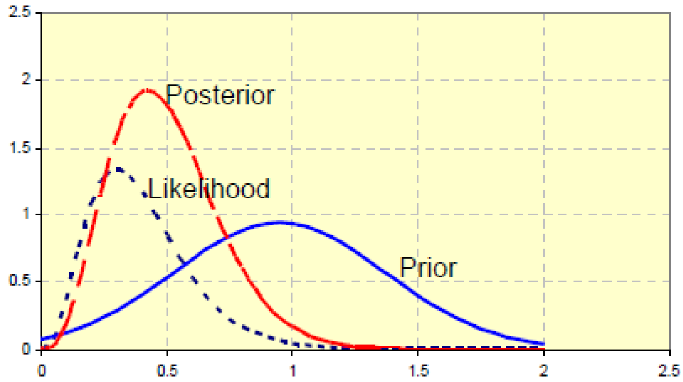

Figure 5.2: **Prior probability density of the failure rate, likelihood of additional sample and posterior probability density for the failure rate.**

For the sake of illustration it is now assumed that a reliability analysis is considered for a new type of pumps, for which no failure data are available. Not knowing better the failure rate for the new type of pumps is represented by the prior probability density for the failure rate for the pump type for which data are available. However, appreciating that the new type of pumps may behave different it is decided to run three experiments on the new type of pumps resulting in the times to failure, given in Table 5.2.

Pump	Time till failure
1	3.2
2	3.5
3	3.3

Table 5.2: **Time till failure for new pumps.**

Assuming that the failure rate is distributed according to the prior probability density function for the failure rate the likelihood function $L(\mathbf{t}|z)$ of the three sample failure times $\mathbf{t} = (t_1, t_2, t_3)^T = (3.2, 3.5, 3.3)^T$ can be calculated from:

$$L(\mathbf{t}|z) = \prod_{i=1}^{3} z \exp(-z x_i) \tag{5.17}$$

which is illustrated in Figure 5.2. The updated probability density function for the uncertain failure rate can be determined using Bayes's rule as:

$$f_Z''(z|\mathbf{t}) = \frac{1}{c} L(\mathbf{t}|z) f_Z'(z) \tag{5.18}$$

where the constant c is determined such that the integral over the posterior probability density equals to one. The rule of Bayes is thus seen to provide a means for combining information of various sources and thus facilitated a combination of subjective information and experiment results in quantitative risk analysis.

From Figure 5.2 it is noticed that whereas the prior probability density for the uncertain failure rate is symmetric (and by the way also allows for realisations in the negative domain!)

the posterior probability density function has been strongly influenced by the Likelihood function and only allows for positive realisations of the failure rate.

Finally in a risk analysis context the failure rates are normally applied for the assessment of the probability of failure for the considered pump type.

Assuming as initially that the times till failure are exponentially distributed the probability that a pump will fail within the time period *T*, for given failure rate z is given by:

$$P_F(T|z) = 1 - \exp(-zT) \tag{5.19}$$

However, as the failure rate is uncertain the probability of failure must be integrated out over the possible realisations of the failure rate weighed with their probabilities, i.e.:

$$P_F(T) = 1 - \int_0^1 \exp(-zT)\, f_Z(z) dz \tag{5.20}$$

thus providing the total unconditional probability of failure. In the present example the probability of failure can be found to be equal to 0.38 taking basis in the posterior probability density function for the failure rate. This compares to a failure probability equal to 0.61 which is found using the prior probability density function.

5.3 Failure rate data for mechanical systems and components

In Table 5.3-5.6 a number of generic data on failure rates are provided based on Stewart and Melchers (1997), for various types of components in the mechanical, electrical and offshore industry. Generic data may serve as a starting point for the analysis of the reliability performance of technical/mechanical components and systems. However, it is very important always to attempt to achieve relevant data for the specific systems and components being subject to analysis. Specific data can then be applied alone, if there is sufficient data to estimate reliable estimates of failure rates, or they may be applied in conjunction with generic data serving as the prior information within the framework of Bayesian updating.

All Modes	Low	Rec	High
Failures/10^6 hours	0.31	1.71	21.94
Failures/10^6 cycles	0.11	0.75	1.51
Repair time (hours)	0.3	0.74	1.3
	Failures/10^6 hours		
Failure mode	Low	Rec	High
Catastrophic	0.13	0.7	9
Zero or maximum output	0.06	0.31	4.05
No change of output with change of input	0.01	0.04	0.45
Functioned without signal	0.03	0.18	2.34
No function with signal	0.03	0.17	2.16
Degraded	0.14	0.75	9.65
Erratic output	0.03	0.17	2.22
High output	0.03	0.15	1.93
Low output	0.01	0.06	0.77
Functioned at improper signal level	0.05	0.29	3.67
Intermitted operation	0.02	0.08	1.06
Incipient	0.04	0.26	3.29

Note: Rec refers to the 'Best estimate'.
Low, High refers to the best and worst data points (i.e. this establishes the range)

Table 5.3: Reliability data for temperature instruments, controls and sensors, Stewart and Melchers (1997) (Source: adapted from IEEE (1984)).

Environmental Stress	**Modifier for failure rate**
High temperature	x 1.75
High radiation	x 1.25
High humidity	x 1.50
High vibration	x 2.00

Table 5.4: Environmental modification factors for temperature instrument, control and sensor reliability data to be multiplied on the failure rates in Table 5.3 depending on the environmental stress. Stewart and Melchers (1997) (Source: adapted from IEEE (1984)).

Population	Samples	Aggregated time in service (10^6hrs)			Number of demands		
17	10	Calendar time 0.3826		Operational time 0.0002	1135		
Failure mode	No. of Failures	Failure rate (per 10^6 hrs)			Repair (man hours)		
		Lower	Mean	Upper	Min.	Mean	Max.
Critical	80 *	120	210	310	-	86	-
	13 **	26000	47000	78000			
Failed to start	75 *	100	190	90	24	86	120
	9 **	6200	32000	69000			
Failed while running	5 *	2	23	51	3	93	130
	4 **	4600	15000	36000			
Degraded	24 *	30	71	120	-	180	-
	3 **	0	14000	45000			
High temperature	22 *	22	66	120	6	190	400
	3 **	0	14000	44000			
Low output	1 *	0.14	2.6	12	-	-	-
Unknown	1 *	0.14	2.6	12	-	96	-
Incipient							
Unknown							
All Modes	303 *	680	840	1000	-	81	-
	45 **	87000	180000	280000			

*Note: *denotes calendar time, ** denotes operational time*

Table 5.5: Reliability data for fire water pumps on offshore platforms, Stewart and Melchers (1997) (Source: adapted from OREDA (1984)).

Component and Failure mode	Unit	Best estimate	Low	High
Electric Motors				
Failure to start	1/D	$3x10^{-4}$	$1x10^{-4}$	$1x10^{-3}$
Failure to run (normal)	1/hrs	$1x10^{-5}$	$3x10^{-6}$	$3x10^{-5}$
Failure to run (extreme environment)	1/hrs	$1x10^{-3}$	$1x10^{-4}$	$1x10^{-2}$
Battery Power systems				
Failure to provide proper output	1/hrs	$3x10^{-6}$	$1x10^{-6}$	$1x10^{-5}$
Switches				
Limit - failure to operate	1/D	$3x10^{-4}$	$1x10^{-4}$	$1x10^{-3}$
Torque - failure to operate	1/D	$1x10^{-4}$	$3x10^{-5}$	$3x10^{-4}$
Pressure - failure to operate	1/D	$1x10^{-4}$	$3x10^{-5}$	$3x10^{-5}$
Manual - fail to transfer	1/D	$1x10^{-5}$	$3x10^{-6}$	$3x10^{-5}$
Contacts short	1/hrs	$1x10^{-7}$	$1x10^{-8}$	$1x10^{-6}$
Pumps				
Failure to start	1/D	$1x10^{-3}$	$3x10^{-4}$	$3x10^{-3}$
Failure to run (normal)	1/hrs	$3x10^{-5}$	$3x10^{-6}$	$3x10^{-4}$
Failure to run (extreme environment)	1/hrs	$1x10^{-3}$	$1x10^{-9}$	$1x10^{-7}$
Valves (motor operated)				
Fails to operate	1/D	$1x10^{-3}$	$3x10^{-4}$	$3x10^{-3}$
Failure to remain open	1/D	$1x10^{-4}$	$3x10^{-5}$	$3x10^{-4}$
External leak or rupture	1/hrs	$1x10^{-8}$	$1x10^{-9}$	$1x10^{-7}$
Circuit breakers				
Failure to operate	1/D	$1x10^{-3}$	$3x10^{-4}$	$3x10^{-3}$
Premature transfer	1/hrs	$1x10^{-6}$	$3x10^{-7}$	$3x10^{-6}$

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Fuses				
Premature, open	1/hrs	1x10^{-6}	3x10^{-7}	3x10^{-6}
Failure to open	1/D	1x10^{-5}	3x10^{-6}	3x10^{-5}
Pipes				
< 75mm, rupture	1/hrs	1x10^{-9}	3x10^{-11}	3x10^{-8}
> 75mm, rupture	1/hrs	1x10^{-10}	3x10^{-12}	3x10^{-9}
Welds				
Leak, containment quality	1/hrs	3x10^{-9}	1x10^{-10}	1x10^{-7}

Table 5.6: Reliability data for mechanical and electrical components. D denotes demand. Stewart and Melchers (1997) (Source: adapted from RSS (1975)).

5.4 Reliability analysis of static components

Concerning the reliability of static components and systems such as structures the situation is different in comparison to that of mechanical and electrical components. For structural components and systems first of all no relevant failure data are available, secondly failures occur significantly more rarely and thirdly the mechanism behind failures is different. Structural failures occur not predominantly due to ageing processes but moreover due to the effect of extreme events, such as e.g. extreme winds, avalanches, snow fall, earthquakes, or combinations hereof.

For the reliability assessment it is therefore necessary to consider the influences acting from the outside i.e. loads and influences acting from the inside i.e. resistances individually. It is thus necessary to establish probabilistic models for loads and resistances including all available information about the statistical characteristics of the parameters influencing these. Such information is e.g. data regarding the annual extreme wind speeds, experiment results of concrete compression strength, etc. These aspects have been treated in a previous chapter. A significant part of the uncertainties influencing the probabilistic modelling of loads and resistances is due to lack of knowledge. Due to that, the failure probabilities, which may be assessed on this basis, must be understood as nominal probabilities, i.e. not reflecting the true probability of failure for the considered structure but rather reflecting the lack of knowledge available about the performance of the structure.

For a structural component for which the uncertain resistance may be modelled by a random variable R with probability density function $f_R(r)$ subjected to the load s the probability of failure P_F may be determined by:

$$P_F = P(R \leq s) = F_R(s) = P(R / s \leq 1) \tag{5.21}$$

In case that also the load is uncertain and modelled by the random variable S with probability density function $f_S(s)$ the probability of failure P_F is:

$$P_F = P(R \leq S) = P(R - S \leq 0) = \int_{-\infty}^{\infty} F_R(x) f_S(x) dx = \int_{-\infty}^{\infty} f_{P_f}(x) dx \tag{5.22}$$

assuming that the load and the resistance variables are statistically independent. This case is called the fundamental case in structural reliability theory. The integration in Equation (5.22) is illustrated in Figure 5.3.

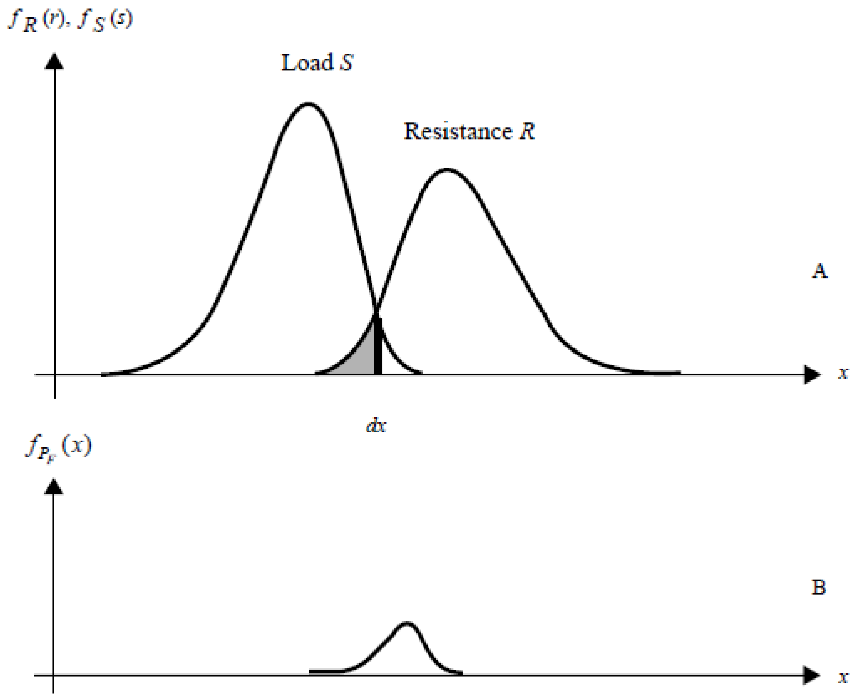

Figure 5.3: A) Illustration of the integration in Equation (5.22) and B) the distribution of the failure probability over the realisations of the resistance *R* and the loading *S*.

In Figure 5.3(A), the contributions to the probability integral of Equation (5.22) are illustrated. Note that the probability of failure is not determined through the overlap of the two curves. In Figure 5.3(B) the integral of Equation (5.22) is illustrated as a function of the realisations of the random variables R and S. The integral of this is not equal to 1 but equal to the failure probability P_F.

There exists no general closed form solution to the integral in Equation (5.22) but for a number of special cases solutions may be derived. One case is when both the resistance variable R and the load variable S are Normal distributed. In this case the failure probability may be assessed directly by considering the random variable M, often referred to as the safety margin:

$$M = R - S \tag{5.23}$$

whereby the probability of failure may be assessed through:

$$P_F = P(R - S \leq 0) = P(M \leq 0) \tag{5.24}$$

where M is also Normal distributed with parameters $\mu_M = \mu_R - \mu_S$ and standard deviation $\sigma_M = \sqrt{\sigma_R^2 + \sigma_S^2}$.

The failure probability may now be determined by use of the standard Normal distribution function as:

$$P_F = \Phi(\frac{0-\mu_M}{\sigma_M}) = \Phi(-\beta) \tag{5.25}$$

where $\mu_M / \sigma_M = \beta$ is called the reliability index. The geometrical interpretation of the *safety index* is illustrated in Figure 5.4.

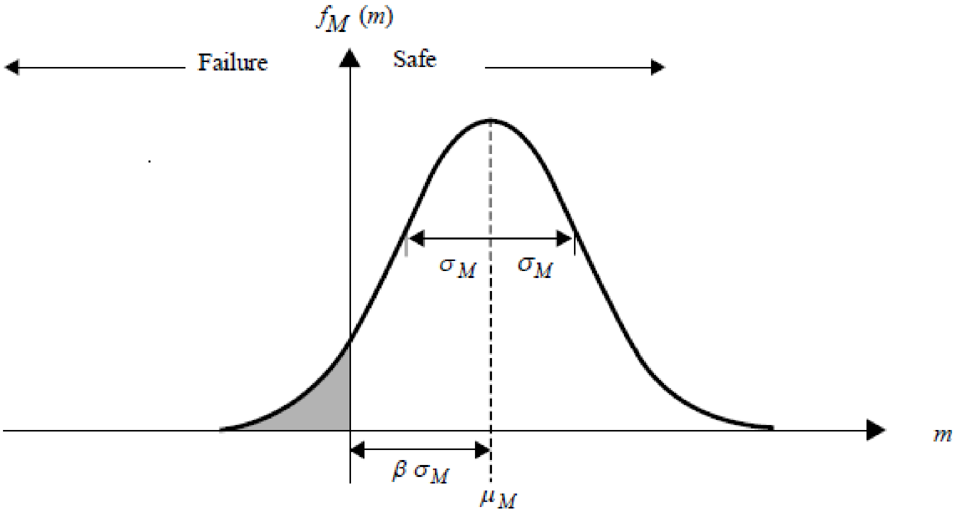

Figure 5.4: Illustration of the probability density function for the Normal distributed safety margin M.

From Figure 5.4 it is seen that the reliability index β is equal to the number of the standard deviation by which the mean value of the safety margin M exceeds zero, or equivalently the distance from the mean value of the safety margin to the most likely failure point.

As indicated previously closed form solutions may also be obtained for other special cases. However, as numerical methods have been developed for the purpose of solving Equation (5.22) these will not be considered in the further.

In the general case the resistance and the load cannot be described by only two random variables but rather by functions of random variables, e.g.:

$$\begin{aligned} R &= f_1(\mathbf{X}) \\ S &= f_2(\mathbf{X}) \end{aligned} \tag{5.26}$$

where $\mathbf{X}$ is a vector with n so-called basic random variables. As indicated in Equation (5.26) both the resistance and the loading may be a function of the same random variables and R and S may thus be statistically dependent.

Furthermore the safety margin

$$M = R - S = f_1(\mathbf{X}) - f_2(\mathbf{X}) = g(\mathbf{X}) \tag{5.27}$$

is in general no longer Normal distributed. The function $g(\mathbf{x})$ is usually denoted the *limit state function*, i.e. an indicator of the state of the considered component. For realisations of the basic random variables $\mathbf{X}$ for which $g(\mathbf{X}) \leq 0$ the component is in a state of failure and otherwise for $g(\mathbf{X}) > 0$ the component is in a safe state.

Setting $g(\mathbf{X}) = 0$ defines a $(n\text{-}1)$ dimensional hyper surface in the space spanned by the n basic random variables. This hyper surface is denoted the failure surface and thus separates all possible realisations $\mathbf{x}$ of the basic random variables $\mathbf{X}$ resulting in failure, i.e. the failure domain, from the realisations resulting in a safe state, the safe domain.

Thereby the probability of failure may be determined through the following n dimensional integral:

$$P_F = \int_{g(\mathbf{x}) \leq 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{5.28}$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function for the vector of basic random variables $\mathbf{X}$ and the integration is performed over the failure domain.

The solution of the integral in Equation (5.28) is by no means a trivial matter except for very special cases and in most practical applications numerical approximate approaches must be pursued. Here it shall, however, be emphasized that usual numerical integration techniques are not appropriate for the solution of the integral in Equation (5.28) due to the fact that the numerical effort to solve it with sufficient accuracy in case of small failure probabilities becomes overwhelming and in addition to this the integration domain is not easy to represent for such algorithms.

This issue shall not be treated further in the present context but deferred to the next chapter describing some of the basics of the so-called methods of structural reliability.