Statistical Mechanics
E0415

Fall 2021, lecture 2
Random walks

Take-home 1

Take-home task 1

Study the material for the week: Sethna Chapter 1. Then, answer the following question(s) and return your answer to MyCourses **deadline on the day before the lecture: September 17 at noon.**

1.t

Have a look at `https://royalsocietypublishing.org/doi/10.1098/rsos.200307` and write a short (1 paragraph) essay commenting on two points: which issues of the book do you recognize in the study, what do you think you would have done in the study?

To the first part of the question:

"Crowd behaviour is a good example of a complex system with a lot of degrees of freedom. This kind of problem is what statistical mechanics was made for."

"Crowds consist of many individuals, and solving for the movement of each of these separately is surely infeasible. Instead, in the article, the crowds were simulated using statistical models, basicall y describing the behaviour of ensembles of people rather than individual people."

"From the issues of the book, the study clearly deals with random walks quite literally since it compares simulated paths of walking people to real ones. The paper also studies instead of individual people the properties of ensemble of relatively randomly smaller systems and try to make sense of them."

More answers

Second part:

"Of course after reading the study it sounds quite clear that of course real humans make more abrupt and rapid changes that in simulations, as those are harder to predict, but would have I been able to realize this if I had not read this study? Maybe, but impossible to really say. "

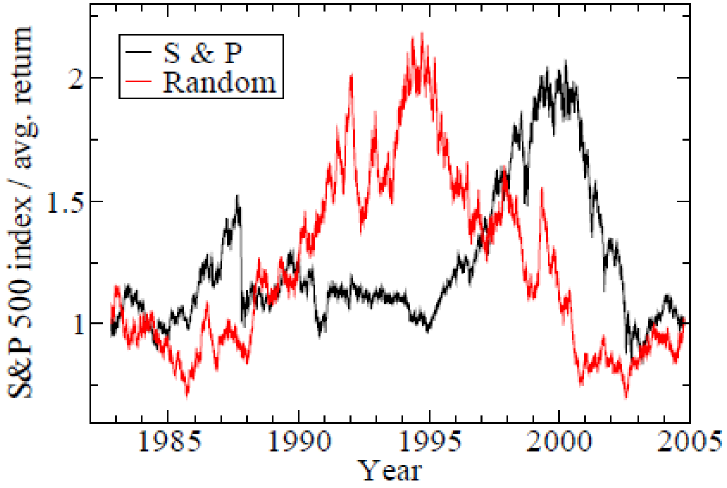
"If I were in charge of the study, I would have additionally tried to apply supervised machine learning methods such as convolutional neural networks to the classified crowds. This would possibly allow generating the most realistic looking crowds. Testing these of course would require classification with the same analysis about random guessing as in the study."

" If I were to have developed the simulation, I would have treated each simulated individual as instances of a data structure, which would carry different parameters that vary from individual to individual. Parameters could include walking speed, destination (a point outside or inside the simulated area), likelihood to meet and interact with friends etc. The statistical distribution of each parameter between simulated individuals could then be appropriately adjusted to match the scenario being simulated."

Universality, scale invariance

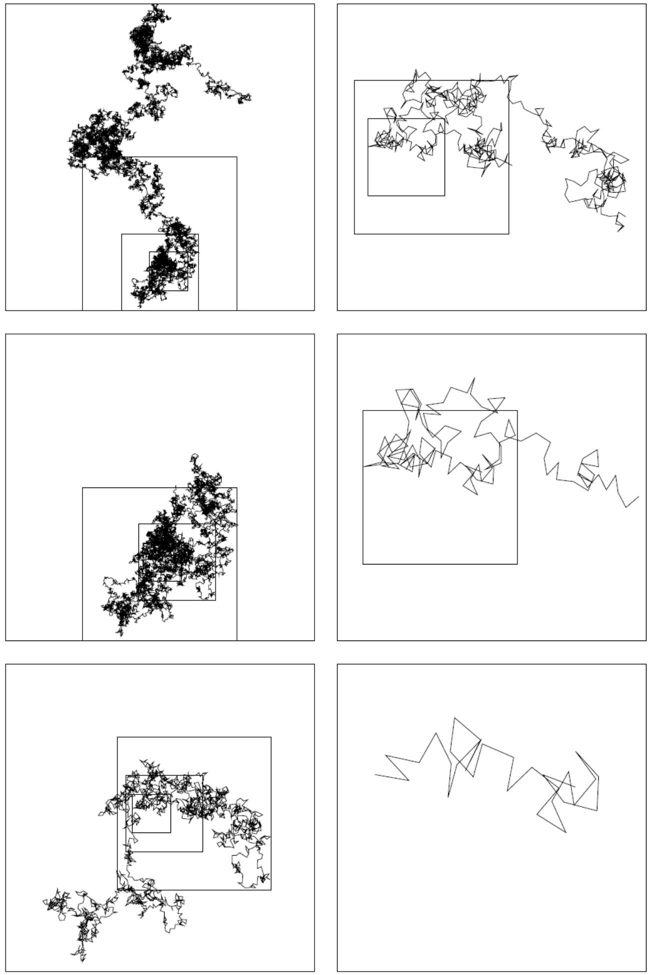
- **Three main points with random walks**: scaling (scale-free) behavior, universality (small details do not matter), probability distributions (and the governing equation(s)).
- Example: coin flips/tosses – do heads or tails win? Square-root law with N.
- Example II: drunkard’s walk (on a 2D plane).
- Universality – compare with polymers (self-avoidance). “Entropic repulsion”, walk exponent becomes ϒ = ¾ (2D), 0.59 (3) – exact and numerical values. “Universal critical exponent” (Self-Avoiding Walks, SAW).

Sorts of RWs



Stock Exchange Index vs.
a simple RW

scale in-
s form a
looks the
same when rescaled. Here each succeeding walk is the first quarter of the previous walk, magnified by a factor of two; the shortest random walk is of length 31, the longest of length 32 000 steps. The left side of Fig. 1.1 is the further evolution of this walk to 128 000 steps.



The Diffusion Equation

An equation for ρ: two interpretations – density of a cloud, pdf of a single RW.

$$\frac{\partial \rho}{\partial t} = D\nabla^2 \rho = D\frac{\partial^2 \rho}{\partial x^2}.$$

Derivation of DE: separation of scales (RW step against the gradient of ρ).

Relation of the diffusion constant D > 0 to the microscopic RW details:

$$D = a^2/2\Delta t.$$

Step size a, timescale Δt.

Currents & external forces

Remember: DE conserves particles, thus the density – "conservation law".

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x}. \qquad J_{\text{diffusion}} = -D\frac{\partial \rho}{\partial x},$$

In the presence of external forces the particles have a deterministic drift.

$$x(t+\Delta t) = x(t) + F\gamma\Delta t + \ell(t).$$

This shows up in the current, and in the equation for ρ.

$$J = \gamma F\rho - D\frac{\partial \rho}{\partial x}.$$

$$\frac{\partial \rho}{\partial t} = -\gamma F\frac{\partial \rho}{\partial x} + D\frac{\partial^2 \rho}{\partial x^2},$$

Case study: density profile with gravity ("atmosphere").

$$\rho^*(x) = A\exp\left(-\frac{\gamma}{D}mgx\right).$$

Solving the diffusion equation

Example: Fourier method. FT in space, substitute: reveals the diffusive timescale and the role of D.

$$\frac{\partial \rho}{\partial t} = \frac{\mathrm{d}\widetilde{\rho}_k}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}kx} = D\frac{\partial^2 \rho}{\partial x^2} = -Dk^2\widetilde{\rho}_k\mathrm{e}^{\mathrm{i}kx},$$
$$\frac{\mathrm{d}\widetilde{\rho}_k}{\mathrm{d}t} = -Dk^2\widetilde{\rho}_k,$$
$$\widetilde{\rho}_k(t) = \widetilde{\rho}_k(0)\mathrm{e}^{-Dk^2t}.$$

General solution as superposition of the FT of the initial profile or condition $\widetilde{\rho_k}(0)$.

$$\rho(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widetilde{\rho}_k(0)\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-Dk^2t}\,\mathrm{d}k.$$

$$\widetilde{\rho}_k(0) = \int_{-\infty}^{\infty}\rho(x,0)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x,$$

Homework

1.2 Waiting times (Sethna 1.3 p. 6) HOMEWORK (5 points)

Take-home task 2

Study the material for the week: Sethna Chapter 2. Then, answer the following question(s) and return your answer to Mycourses **deadline on the day before the lecture: September 30 at noon**.

2.t

Let us turn this into an exercise in gambling. You play heads and tails (toss a coin, and guess the outcome: win or lose the coin). Three questions: you start with 10 coins. Give an argument how the distribution of times it takes for you to lose all your coins looks like. What happens if you play till you have zero, or until you won all the 10 coins of your friend? Let us now consider the case where the coin is not fair: the fractional Brownian motion, where the subsequent outcomes are correlated (positively or negatively). How does that influence qualitatively those outcomes?

Projects, presentation

Wisa Förbom, Into Pääkkönen, Heikki Systä - project 1, paper 8

Markus Lehtisalo, Laura Utriainen, Amresh Thakur - project 2, paper 2

Sara Tuomaala, Fredrik Ihamuotila, Reko Penttilä - project 3, paper 3

Minh Pham Nguyen, Mikko Malmi, Atte Tupala - project 4, paper 7

Senna Luntama, Kerttu Aronen, Antti Karjasilta - project 5, paper 1

Daniel Rapinoja de Carvalho, Juan Pablo Rubio Perez, He Jingwang - project 6, paper 6

Rita Santos, Yanis Le Fur, Niko Savola - project 7, paper 5

Use the course Zoom to agree with your group (and maybe to meet)

Other scheduling

Project “show”; probably 7th of Dec.

Paper presentations: next slide

Friday lectures: we usually do 13.15

Presentations

8.10. paper 1: Negative temperatures (Senna Luntama, Kerttu Aronen, Antti Karjasilta)

15.10. paper 2: Physics & single cell biology (Markus Lehtisalo, Laura Utriainen, Amresh Thakur)

22.10. paper 5: Negative Representation and Instability in Democratic Elections (Rita Santos, Yanis Le Fur, Niko Savola)

5.11. paper 7: Entropy production (Minh Pham Nguyen, Mikko Malmi, Atte Tupala)

12.11. paper 6: Quantum phase transition (Daniel Rapinoja de Carvalho, Juan Pablo Rubio Perez, He Jingwang)

19.11. paper 3: Non-equilibrium transition & Game of Life (Sara Tuomaala, Fredrik Ihamuotila, Reko Penttilä)

26.11. paper 8: Avalanches and their shape (Wisa Förbom, Into Pääkkönen, Heikki Systä)