Aalto University
School of Electrical Engineering

ELEC-E4130 Electromagnetic fields, Autumn 2021

Plane electromagnetic waves

Henrik Wallén

October 14, 2021

Plane electromagnetic waves

Outline for lecture weeks 4–5

Week 4: Plane wave propagation in unbounded homogeneous media

- Plane waves in lossless media [8–2]
- Lossy media [8–3]
- Group velocity [8–4]
- Power flow [8–5]

Week 5: Reflection and transmission at planar interfaces

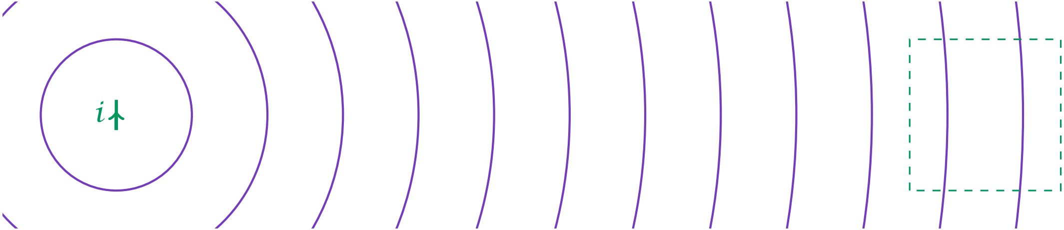
- Normal incidence at PEC [8–6]
- Dielectric boundary [8–8]
- Multiple dielectric layers [8–9]
- Oblique incidence at dielectric boundary (or PEC) [8–10, 8–7]

D.K. Cheng, *Field and Wave Electromagnetics*, 2nd Ed., Chapter 8

Introduction

Last week, we saw that the solutions of Maxwell's equations in free space are electromagnetic waves traveling with velocity $c = 1/\sqrt{\mu_0 \varepsilon_0}$ = speed of light.

A small source (antenna) creates a spherical wave



that locally, sufficiently far away can be approximated with a **uniform plane wave**, where $\mathbf{E}$ has the same direction, amplitude, and phase in infinite planes perpendicular to the direction of propagation (similarly for $\mathbf{H}$).

Plane waves are mathematically much easier than spherical waves.

Plane waves in lossless media

Cheng 8-2

Plane-wave solution

In homogeneous non-conducting source-free media, the time-harmonic electric field (vector phasor $\mathbf{E}$) satisfies the homogeneous Helmholtz' equation

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = \mathbf{0}, \qquad k = \omega\sqrt{\mu\varepsilon} = \text{wavenumber}.$$

If we assume that $\mathbf{E} = \mathbf{E}(z)$, this simplifies to

$$\frac{d^2\mathbf{E}}{dz^2} + k^2\mathbf{E} = \mathbf{0} \qquad \Rightarrow \qquad \mathbf{E}(z) = \mathbf{E}^+e^{-jkz} + \mathbf{E}^-e^{+jkz},$$

where $\mathbf{E}^\pm$ are some complex constant vectors. From Gauss' law follows

$$\nabla\cdot\mathbf{D} = 0 \quad \Rightarrow \quad \frac{\partial E_z}{\partial z} = 0 \quad \Rightarrow \quad \mathbf{a}_z\cdot\mathbf{E}^\pm = 0.$$

At this point, we choose $\boxed{\mathbf{E} = \mathbf{a}_x E_0 e^{-jkz}}$ with $E_0 > 0$.

Phase velocity and wavelength

The corresponding time-dependent electric field is

$$\mathbf{E}(t) = \mathscr{Re}\left[\mathbf{a}_x E_0 e^{-jkz} e^{j\omega t}\right] = \mathbf{a}_x E_0 \cos(\omega t - kz).$$

This is a sinusoidal wave traveling in the $+z$ direction:

$$\omega t - kz = A = \text{constant} \quad \Rightarrow \quad \boxed{u_p = \frac{dz}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \textbf{phase velocity}}$$

Looking at the field at a fixed time (e.g. $t = 0$) we see that the **wavelength** is

$$\boxed{\lambda = \frac{2\pi}{k}}$$

Similarly, for a fixed place (e.g. $z = 0$) we get the **period** $T = 2\pi/\omega = 1/f$, where ω = **angular frequency** and f = **frequency**.

Magnetic field of the plane wave

Using Faraday's law, we get

$$
\begin{aligned}
\mathbf{H} &= \frac{1}{-j\omega\mu}\nabla\times\mathbf{E} = \frac{j}{\omega\mu}\begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & 0 & 0\end{vmatrix} \\
&= \frac{j}{\omega\mu}\left[\mathbf{a}_y\frac{\partial}{\partial z}\left(E_0\, e^{-jkz}\right)\right] = \mathbf{a}_y\frac{k}{\omega\mu}E_0\, e^{-jkz} = \mathbf{a}_y\frac{E_0}{\eta}e^{-jkz},
\end{aligned}
$$

where $\eta = \omega\mu/k = \sqrt{\mu/\varepsilon}$ is the **intrinsic impedance** of the medium.

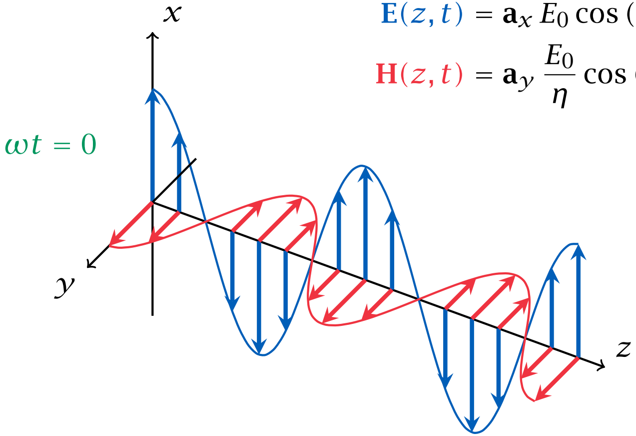
The fields are thus in-phase and orthogonal at every point and every time:

$$
\begin{aligned}
\mathbf{E}(t) &= \mathbf{a}_x\, E_0 \cos\left(\omega t - kz\right) \\
\mathbf{H}(t) &= \mathbf{a}_y\, \frac{E_0}{\eta} \cos\left(\omega t - kz\right)
\end{aligned}
$$

EM fields of the plane wave

$$\mathbf{E}(z,t) = \mathbf{a}_x E_0 \cos(\omega t - kz)$$

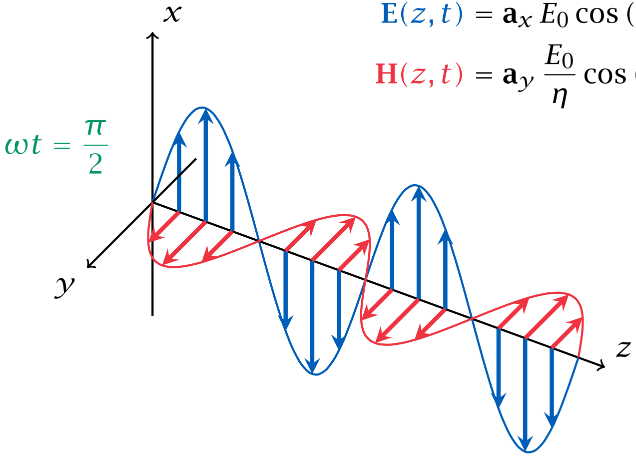
$$\mathbf{H}(z,t) = \mathbf{a}_y \frac{E_0}{\eta} \cos(\omega t - kz)$$



EM fields of the plane wave

$$\mathbf{E}(z,t) = \mathbf{a}_x E_0 \cos(\omega t - kz)$$

$$\mathbf{H}(z,t) = \mathbf{a}_y \frac{E_0}{\eta} \cos(\omega t - kz)$$



Example problem 4.1: plane wave in $-x$ direction

Express the vector phasors $\mathbf{E}(x)$ and $\mathbf{H}(x)$ for a homogeneous plane wave propagating in the negative x direction, when the electric field is y directed with amplitude $E_0 > 0$.

Solution: Comparing the given description with the plane wave on the previous slides, we can write the electric field

$$\mathbf{E}(x) = \mathbf{a}_y E_0 \, e^{+jkx}.$$

Using Faraday's law, we get the magnetic field

$$\mathbf{H}(x) = \frac{1}{-j\omega\mu} \nabla \times \mathbf{E}(x) = \frac{j}{\omega\mu} \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}$$

$$= \frac{j}{\omega\mu} \left[\mathbf{a}_z \frac{\partial}{\partial x} \left(E_0 \, e^{+jkx} \right) \right] = \mathbf{a}_z \frac{-k}{\omega\mu} E_0 \, e^{+jkx} = -\mathbf{a}_z \frac{E_0}{\eta} e^{+jkx}.$$

Plane-wave equations

To derive a more general plane-wave solution, assume that $\mathbf{E}$ and $\mathbf{H}$ both have the spatial dependence

$$\boxed{e^{-j\mathbf{k}\cdot\mathbf{R}}} \qquad \begin{cases} \mathbf{k} = \text{wavenumber vector (or wavevector)} \\ \mathbf{R} = \text{position vector} \end{cases}$$

Then we have

$$\nabla e^{-j\mathbf{k}\cdot\mathbf{R}} = \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) e^{-j(k_x x + k_y y + k_z z)} = -j\mathbf{k}\, e^{-j\mathbf{k}\cdot\mathbf{R}}$$

and we can replace $\boxed{\nabla \longrightarrow -j\mathbf{k}}$ in Maxwell's equations:

$$\begin{cases} \nabla \times \mathbf{E} = -j\omega\mu\,\mathbf{H} \\ \nabla \times \mathbf{H} = j\omega\varepsilon\,\mathbf{E} \end{cases} \longrightarrow \boxed{\begin{matrix} \mathbf{k} \times \mathbf{E} = \omega\mu\,\mathbf{H} \\ \mathbf{k} \times \mathbf{H} = -\omega\varepsilon\,\mathbf{E} \end{matrix}}$$

For plane waves, we thus have two very simple algebraic equations.

Solution of the plane-wave equations

From the plane wave equations

$$\mathbf{k} \times \mathbf{E} = \omega\mu\,\mathbf{H}, \quad \mathbf{k} \times \mathbf{H} = -\omega\varepsilon\,\mathbf{E}$$

follows that $\mathbf{k} \cdot \mathbf{E} = \mathbf{k} \cdot \mathbf{H} = 0$, that $\mathbf{E} \times \mathbf{H} \upuparrows \mathbf{k}$, and that the magnitude of the wavevector is the wavenumber:

$$\begin{aligned} \mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = \mathbf{k} \times (\omega\mu\,\mathbf{H}) \quad &\Leftrightarrow \quad \mathbf{k}\,(\mathbf{k} \cdot \mathbf{E}) - \mathbf{E}\,(\mathbf{k} \cdot \mathbf{k}) = -\omega^2\mu\varepsilon\mathbf{E} \\ &\Leftrightarrow \quad \mathbf{k} \cdot \mathbf{k} = k^2 = \omega^2\mu\varepsilon \end{aligned}$$

Denoting $\mathbf{k} = k\mathbf{a}_n$ we can write the plane wave equations and their solution as

$$\boxed{\begin{aligned} \mathbf{H} &= \frac{1}{\eta}\,\mathbf{a}_n \times \mathbf{E} \\ \mathbf{E} &= -\eta\,\mathbf{a}_n \times \mathbf{H} \end{aligned}} \qquad \boxed{\begin{aligned} \mathbf{E} &= \mathbf{E}_0\, e^{-jk\mathbf{a}_n \cdot \mathbf{R}}, \quad \mathbf{a}_n \cdot \mathbf{E}_0 = 0 \\ \mathbf{H} &= \frac{1}{\eta}\,(\mathbf{a}_n \times \mathbf{E}_0)\, e^{-jk\mathbf{a}_n \cdot \mathbf{R}} \end{aligned}}$$

The big picture

Assumptions

- Time-harmonic fields $\left(e^{j\omega t}\right)$
- Source-free homogeneous medium: $\rho = 0, \mathbf{J} = \mathbf{0}$ and $\varepsilon, \mu =$ constants
- Spatial dependence $\left(e^{-j\mathbf{k}\cdot\mathbf{R}}\right)$

Mathematically, this corresponds to the Fourier transform of Maxwell's equations in both time and space.

Maxwell's equations in differential form

$\downarrow \quad \dfrac{\partial}{\partial t} \longrightarrow j\omega$

Time-harmonic Maxwell's equations

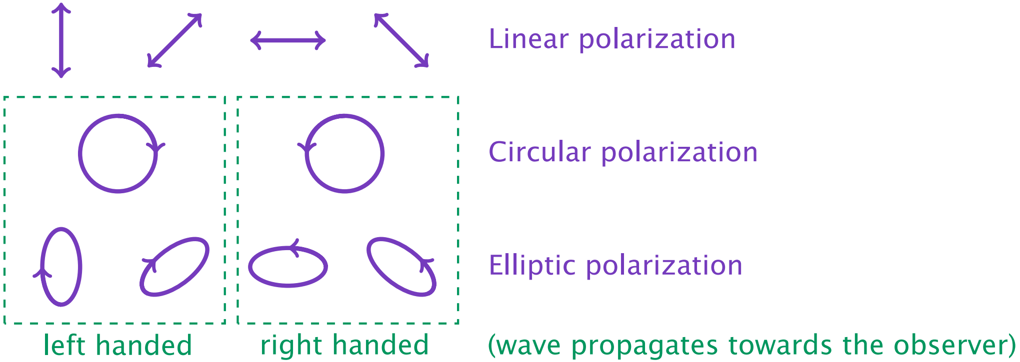
$\downarrow \quad \nabla \longrightarrow -j\mathbf{k}$

Plane-wave equations

losses: $\varepsilon \longrightarrow \varepsilon_c = \varepsilon - j\dfrac{\sigma}{\omega}$

Polarization of plane waves

The **polarization** of a uniform plane wave is defined by the time-varying behavior of $\mathbf{E}(t)$ at a fixed position.



Polarization of plane waves

A homogeneous plane wave propagating in the $+z$ direction in lossless media can be expressed as

$$\mathbf{E} = \left(\mathbf{a}_x\, E_{x0} + \mathbf{a}_y\, E_{y0}\right) e^{-jkz} = (\mathbf{E}_{\mathrm{re}} + j\,\mathbf{E}_{\mathrm{im}})\, e^{-jkz},$$

where the E_{x0}, E_{y0} are complex constants and the vectors $\mathbf{E}_{\mathrm{re}}, \mathbf{E}_{\mathrm{im}}$ are real and $\perp \mathbf{a}_z$.

Linear polarization

The polarization in linear, if the x and y components E_{x0} and E_{y0} are in-phase or in opposite phase, or equivalently, if $\mathbf{E}_{\mathrm{re}} \parallel \mathbf{E}_{\mathrm{im}}$.

Circular polarization

The plane wave is circularly polarized, if

$$\boxed{|\mathbf{E}_{\mathrm{re}}| = |\mathbf{E}_{\mathrm{im}}|\,, \quad \mathbf{E}_{\mathrm{re}} \perp \mathbf{E}_{\mathrm{im}}.}$$

Right-hand circularly polarized (RCP) plane wave:

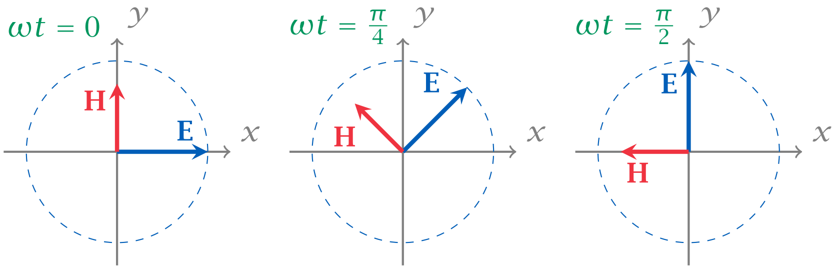
$$\mathbf{E} = \left(\mathbf{a}_x - j\,\mathbf{a}_y\right) E_0\, e^{-jkz} \qquad\qquad (E_0, k > 0)$$

$$\begin{aligned}
\Leftrightarrow \quad \mathbf{E}(t) &= \mathcal{R}e\left\{\left(\mathbf{a}_x - j\,\mathbf{a}_y\right) E_0\, e^{-jkz} e^{+j\omega t}\right\} \\
&= E_0\, \mathcal{R}e\left\{\left(\mathbf{a}_x - j\,\mathbf{a}_y\right) [\cos(\omega t - kz) + j\sin(\omega t - kz)]\right\} \\
&= E_0\left[\mathbf{a}_x \cos(\omega t - kz) + \mathbf{a}_y \sin(\omega t - kz)\right]
\end{aligned}$$

The magnetic field phasor is $\mathbf{H} = \dfrac{1}{\eta}\mathbf{a}_z \times \mathbf{E}$.

Circular polarization (RCP)

Electric and magnetic fields in the xy plane ($z = 0$) as function of time:



Both **E** and **H** have constant amplitude and rotate counterclockwise. The wave is traveling towards the viewer (in the $+z$ direction).

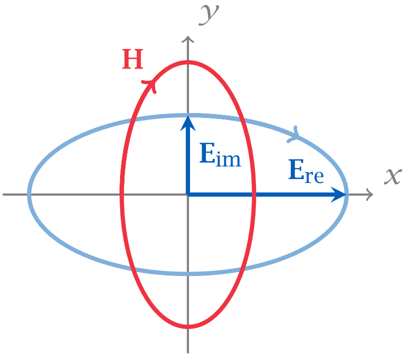
If we replace $(\mathbf{a}_x - j\,\mathbf{a}_y) \longrightarrow (\mathbf{a}_x + j\,\mathbf{a}_y)$ in the electric field phasor, we get the corresponding **left-hand circular polarization** (LCP).

Elliptic polarization

Left-hand elliptic polarization with axis ratio 2

$$\mathbf{E} = \left(2\mathbf{a}_x + j\,\mathbf{a}_y\right) E_0 e^{-jkz}, \quad \mathbf{H} = \left(2\mathbf{a}_y - j\,\mathbf{a}_x\right) \frac{E_0}{\eta} e^{-jkz}$$

The ellipse is easy to draw if $\mathbf{E}_{\mathrm{re}} \perp \mathbf{E}_{\mathrm{im}}$:



- The direction of rotation can be found by turning $\mathbf{E}_{\mathrm{im}} \to \mathbf{E}_{\mathrm{re}}$ the shorter way.
- The ellipse drawn by $\mathbf{H}$ has the same shape, but it is turned $90°$.
- $\mathbf{E}(t) \perp \mathbf{H}(t)$ everywhere at every time, when the medium is lossless.

Example problem 4.2: polarization

Study the following two conditions and determine how they are related to the polarization of a plane wave:

(a) $\mathbf{E} \cdot \mathbf{E} = 0$ **(b)** $\mathbf{E} \times \mathbf{E}^* = 0$

(A non-zero vector phasor $\mathbf{E}$ can satisfy one of the conditions, but not both conditions simultaneously.)

Solution: For $\mathbf{E} = (\mathbf{E}_{\rm re} + j\mathbf{E}_{\rm im})\, e^{-jkz}$, we get

$$\begin{aligned} \mathbf{E} \cdot \mathbf{E} = 0 \quad &\Leftrightarrow \quad |\mathbf{E}_{\rm re}|^2 - |\mathbf{E}_{\rm im}|^2 + 2j\mathbf{E}_{\rm re} \cdot \mathbf{E}_{\rm im} = 0 \\ &\Leftrightarrow \quad |\mathbf{E}_{\rm re}| = |\mathbf{E}_{\rm im}| \text{ and } \mathbf{E}_{\rm re} \perp \mathbf{E}_{\rm im} \quad \Leftrightarrow \quad \text{circular pol.} \end{aligned}$$

$$\begin{aligned} \mathbf{E} \times \mathbf{E}^* = 0 \quad &\Leftrightarrow \quad (\mathbf{E}_{\rm re} + j\mathbf{E}_{\rm im}) \times (\mathbf{E}_{\rm re} - j\mathbf{E}_{\rm im}) = -2j\mathbf{E}_{\rm re} \times \mathbf{E}_{\rm im} = 0 \\ &\Leftrightarrow \quad \mathbf{E}_{\rm re} \times \mathbf{E}_{\rm im} = 0 \quad \Leftrightarrow \quad \mathbf{E}_{\rm re} \parallel \mathbf{E}_{\rm im} \quad \Leftrightarrow \quad \text{linear pol.} \end{aligned}$$

Plane waves in lossy media

Cheng 8-3

Plane wave in lossy media

In lossy media, the complex permittivity $\varepsilon_c = \varepsilon' - j\varepsilon'' = \varepsilon - j\sigma/\omega$ makes the wavenumber complex and we can express a plane wave propagating in the $+z$ direction as

$$\mathbf{E} = \mathbf{a}_x E_0 e^{-\gamma z} = \mathbf{a}_x \underbrace{E_0 e^{-\alpha z}}_{\text{amplitude}} \underbrace{e^{-j\beta z}}_{\text{phase}}, \qquad \gamma = \alpha + j\beta = j\omega\sqrt{\mu\varepsilon_c},$$

where

$$\boxed{\begin{aligned} \gamma &= \textbf{propagation constant} = jk_c \\ \alpha &= \textbf{attenuation constant} > 0 \\ \beta &= \textbf{phase constant} > 0 \end{aligned}}$$

Simlilarly, also the intrinsic impedance is complex $\eta_c = \sqrt{\mu/\varepsilon_c}$.

Special cases

In an **ideal dielectric** ($\varepsilon'' = 0$), the wave propagates without attenuation ($\alpha = 0$, $\beta = k$).

In a **low-loss dielectric** or good insulator ($\varepsilon'' \ll \varepsilon'$), the wave propagates with small attenuation.

In a **good conductor** ($\varepsilon'' \gg \varepsilon'$), the wave attenuates quickly.

In a **perfect conductor** ($\varepsilon'' = \infty$), there cannot be any waves.

Low-loss dielectrics ($\varepsilon''/\varepsilon' \ll 1$)

Using the approximation

$$\sqrt{1-x} \approx 1 - x/2 - x^2/8, \qquad |x| \ll 1,$$

we get

$$\begin{aligned}\gamma = j\omega\sqrt{\mu\varepsilon_c} = j\omega\sqrt{\mu(\varepsilon' - j\varepsilon'')} &= j\omega\sqrt{\mu\varepsilon'}\sqrt{1 - j\frac{\varepsilon''}{\varepsilon'}} \\ &\approx j\omega\sqrt{\mu\varepsilon'}\left[1 - j\frac{\varepsilon''}{2\varepsilon'} + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] = \alpha + j\beta.\end{aligned}$$

$$\Rightarrow \quad \boxed{\begin{aligned}\alpha &\approx \frac{\omega\varepsilon''}{2}\sqrt{\frac{\mu}{\varepsilon'}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon'}} \\ \beta &\approx \omega\sqrt{\mu\varepsilon'}\left[1 + \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] \approx \omega\sqrt{\mu\varepsilon'}\end{aligned}}$$

So β is about the same as in a lossless medium and $\alpha/\beta \approx \varepsilon''/(2\varepsilon') \ll 1$.

Low-loss dielectrics (cont.)

Similar approximations for the intrinsic impedance and phase velocity give

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon'}}\left(1 - j\frac{\varepsilon''}{\varepsilon'}\right)^{-1/2} \approx \sqrt{\frac{\mu}{\varepsilon'}}\left(1 + j\frac{\varepsilon''}{2\varepsilon'}\right)$$

$$u_p = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{\mu\varepsilon'}}\left[1 - \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2\right] \approx \frac{1}{\sqrt{\mu\varepsilon'}}$$

The (relatively small) imaginary part of η_c creates a phase difference between $\mathbf{E}$ and $\mathbf{H}$.

If the loss tangent $\tan\delta_c = \varepsilon''/\varepsilon' = 0.1$, we have

$$1 \pm \frac{1}{8}\left(\frac{\varepsilon''}{\varepsilon'}\right)^2 = 1 \pm \frac{1}{800} \approx 1 \pm 1‰$$

so the second approximation ($\approx$) in β and u_p seems fairly good.

Good conductors ($\varepsilon'' \gg \varepsilon'$)

Using the (somewhat crude?) approximation $\varepsilon_c \approx -j\varepsilon''$, we get

$$\begin{aligned} \gamma^2 &= -\omega^2\mu\varepsilon_c \approx j\omega^2\mu\varepsilon'' \\ &= (\alpha + j\beta)^2 = \alpha^2 + j2\alpha\beta - \beta^2 \end{aligned} \quad \Leftrightarrow \quad \begin{cases} \alpha \approx \beta \\ \omega^2\mu\varepsilon'' \approx 2\alpha\beta \end{cases}$$

$$\Leftrightarrow \quad \alpha \approx \beta \approx \sqrt{\frac{\omega^2\mu\varepsilon''}{2}} = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f \mu\sigma}$$

and

$$\eta_c = \sqrt{\frac{\mu}{\varepsilon_c}} \approx \sqrt{j\frac{\mu}{\varepsilon''}} = (1+j)\sqrt{\frac{\mu}{2\varepsilon''}} = (1+j)\sqrt{\frac{\omega\mu}{2\sigma}}.$$

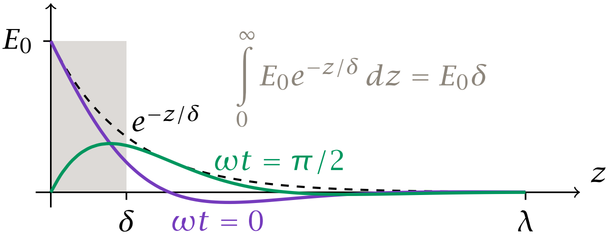
$$\boxed{\alpha \approx \beta \approx \sqrt{\pi f\mu\sigma}, \qquad \eta_c \approx (1+j)\frac{\alpha}{\sigma}} \qquad u_p = \frac{\omega}{\beta} \approx \sqrt{\frac{2\omega}{\mu\sigma}}$$

Good conductors (cont.)

For good conductors, we often use the **skin depth** $\delta = 1/\alpha$, which is the distance through which the amplitude decreases with a factor $1/e \approx 0.368$.

In a good conductor, $\dfrac{2\pi}{\lambda} = \beta \approx \alpha = \dfrac{1}{\delta} \quad \Rightarrow \quad \delta \approx \dfrac{\lambda}{2\pi}$

$$E(z,t) = E_0\, e^{-z/\delta} \cos(\omega t - z/\delta)$$



Complex intrinsic impedance $\eta_c = |\eta_c| \underline{/45^\circ} \Rightarrow \mathbf{H}$ field lags 45° behind the $\mathbf{E}$ field.

Group velocity

Cheng 8-4

Sum of two time-harmonic waves

As earlier discussed, the wave $E_x(t) = E_0 \cos(\omega t - \beta z)$ travels with **phase velocity** u_p in the $+z$ direction:

$$\omega t - \beta z = \text{constant} \quad \Rightarrow \quad u_p = \frac{\partial z}{\partial t} = \frac{\omega}{\beta}$$

Let us now sum two waves with slightly different angular frequencies $\omega \pm \Delta\omega$ and phase constants $\beta \pm \Delta\beta$:

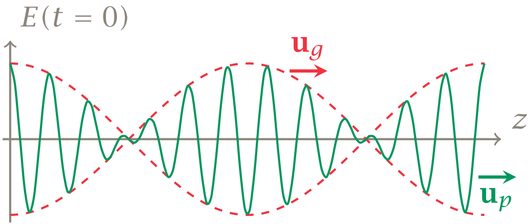
$$\begin{aligned} E(t) = \quad & E_0 \cos\left[(\omega + \Delta\omega)\, t - (\beta + \Delta\beta)\, z\right] \\ & + E_0 \cos\left[(\omega - \Delta\omega)\, t - (\beta - \Delta\beta)\, z\right] \\ = \quad & E_0 \cos\left[(\omega t - \beta z) + (\Delta\omega t - \Delta\beta z)\right] \\ & + E_0 \cos\left[(\omega t - \beta z) - (\Delta\omega t - \Delta\beta z)\right] \end{aligned}$$

Using the formula $\cos(x + y) + \cos(x - y) = 2\cos(x)\cos(y)$, we get

$$E(t) = 2E_0 \cos(\omega t - \beta z)\cos(\Delta\omega t - \Delta\beta z)$$

Propagation velocities

$$E(t) = 2E_0 \underbrace{\cos(\omega t - \beta z)}_{\text{rapid osc.}} \underbrace{\cos(\Delta\omega t - \Delta\beta z)}_{\text{envelope}}$$



The rapidly oscillating wave inside the envelope travels with velocity

$$u_p = \frac{\omega}{\beta}$$

while the envelope travels with velocity

$$u_g = \frac{\Delta\omega}{\Delta\beta}.$$

In the limit $\Delta\omega \to 0$, we get the **group velocity** $\boxed{v_g = \frac{1}{\partial\beta/\partial\omega}}$

The modulation, wave packets, and information coded in a narrow-band signal propagates with this speed.

Phase and group velocity & dispersion

In lossless ordinary media the phase and group velocities are the same for all frequencies

$$\beta = \omega\sqrt{\mu\varepsilon} \quad \Rightarrow \quad u_p = u_g = \frac{1}{\sqrt{\mu\varepsilon}},$$

which implies that any wave or signal will travel with intact shape.

Dispersion

If the material properties depend on the frequency, we get different $u_p(\omega)$ and $u_g(\omega)$ and so the shape of a traveling wave will be distorted. This phenomenon is called **dispersion** and a lossy dielectric is an example of a **dispersive material**.

A lossy material is in principle always dispersive, since $\varepsilon_c = \varepsilon - j\sigma/\omega$, but the dispersion can often be neglected if the losses are small enough.

Example problem 4.3: dispersion in lossless plasma

The earth's ionosphere is approximately a medium with $\mu = \mu_0$ and

$$\varepsilon = \varepsilon_0 \left(1 - \frac{\omega_p^2}{\omega^2}\right) = \varepsilon_0 \left(1 - \frac{f_p^2}{f^2}\right)$$

where the plasma frequency $0.9\,\mathrm{MHz} < f_p < 9\,\mathrm{MHz}$. (Radio waves with $f < f_c$ cannot propagate in the ionosphere and they will be reflected back to earth.)

Determine the phase velocity u_p and group velocity u_g for waves with $f > f_p$. Do you notice any interesting relations between u_p, u_g, and c?

Solution

Using the propagation constant

$$\beta = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}}$$

gives

$$u_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} > c$$

$$u_g = \left(\frac{\partial\beta}{\partial\omega}\right)^{-1} = \left(\frac{1}{c}\sqrt{1 - \frac{\omega_p^2}{\omega^2}} + \frac{\omega}{c}\frac{\omega_p^2/\omega^3}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}\right)^{-1} = c\sqrt{1 - \frac{\omega_p^2}{\omega^2}} < c$$

and $u_p u_g = c^2$. (We will later see the same dispersion for guided waves in an empty PEC waveguide.)

Power flow and the Poynting vector

Cheng 8-5

Poynting's theorem

Let us for a moment go back to time-dependent fields and study the divergence of $\mathbf{E} \times \mathbf{H}$:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H}) = -\mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right)$$

If we assume a simple medium with constant ε, μ, σ and notice that

$$\frac{\partial}{\partial t} \left(\varepsilon E^2 \right) = 2\varepsilon E \frac{\partial E}{\partial t} = 2\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}$$

we get

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = -\frac{\partial}{\partial t} \left(\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) - \sigma E^2.$$

Integrating over a volume V with surface S and applying the divergence theorem gives **Poynting's theorem**

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \varepsilon E^2 + \frac{1}{2} \mu H^2 \right) dv - \int_V \sigma E^2 \, dv$$

Poynting's theorem and the Poynting vector

We can interpret Poynting's theorem

$$\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = -\frac{\partial}{\partial t} \int_V \left(\frac{1}{2}\varepsilon E^2 + \frac{1}{2}\mu H^2 \right) dv - \int_V \sigma E^2 \, dv$$

as power leaving the volume = decrease in stored EM energy - ohmic losses.

The **Poynting vector**

$$\boxed{\mathscr{P}(t) = \mathbf{E}(t) \times \mathbf{H}(t)} \qquad [\mathscr{P}] = \mathrm{W/m^2}$$

or power density vector is the instantaneous power flow (magnitude and direction) per unit area. Integrating the Poynting vector over a surface gives the instantaneous total power flowing through the surface.

The Poynting vector for time-harmonic fields

For complex numbers $z = a + jb$,

$$\mathscr{Re}[z] = \frac{1}{2}(z + z^*)\,, \qquad \text{where } z^* = a - jb.$$

The Poynting vector of a time-harmonic EM field is

$$\begin{aligned}
\boldsymbol{\mathscr{P}} &= \mathbf{E}(t) \times \mathbf{H}(t) = \mathscr{Re}\left[\mathbf{E}\,e^{j\omega t}\right] \times \mathscr{Re}\left[\mathbf{H}\,e^{j\omega t}\right] \\
&= \frac{1}{2}\left(\mathbf{E}\,e^{j\omega t} + \mathbf{E}^* e^{-j\omega t}\right) \times \frac{1}{2}\left(\mathbf{H}\,e^{j\omega t} + \mathbf{H}^* e^{-j\omega t}\right) \\
&= \frac{1}{2}\,\frac{1}{2}\left(\mathbf{E}\times\mathbf{H}\,e^{j2\omega t} + \mathbf{E}\times\mathbf{H}^* + \mathbf{E}^*\times\mathbf{H} + \mathbf{E}^*\times\mathbf{H}^* e^{-j2\omega t}\right) \\
&= \frac{1}{2}\mathscr{Re}\left[\mathbf{E}\times\mathbf{H}^*\right] + \frac{1}{2}\mathscr{Re}\left[\mathbf{E}\times\mathbf{H}\,e^{j2\omega t}\right]
\end{aligned}$$

= average power density + something with double frequency

Average power density

In many cases the average power density

$$\boxed{\mathcal{P}_{\text{av}} = \frac{1}{2}\,\mathcal{Re}\left[\mathbf{E}\times\mathbf{H}^*\right]}$$

is a more relevant quantity than the instantaneous Poynting vector.

Example: For a **plane wave in lossy media**

$$\mathbf{E} = \mathbf{a}_x E_0 e^{-\alpha z} e^{-j\beta z}, \qquad \alpha + j\beta = j\omega\sqrt{\mu\varepsilon_c}$$
$$\mathbf{H} = \mathbf{a}_y \frac{E_0}{\eta} e^{-\alpha z} e^{-j\beta z}, \qquad \eta = \sqrt{\mu/\varepsilon_c} = |\eta|\, e^{j\theta_\eta}$$

the average power density is

$$\mathcal{P}_{\text{av}} = \frac{1}{2}\,\mathcal{Re}\left[\left(\mathbf{a}_x E_0 e^{-\alpha z} e^{-j\beta z}\right)\times\left(\mathbf{a}_y \frac{E_0^*}{|\eta|\, e^{-j\theta_\eta}} e^{-\alpha z} e^{+j\beta z}\right)\right]$$
$$= \mathbf{a}_z \frac{|E_0|^2}{2\,|\eta|} e^{-2\alpha z}\cos(\theta_\eta)$$

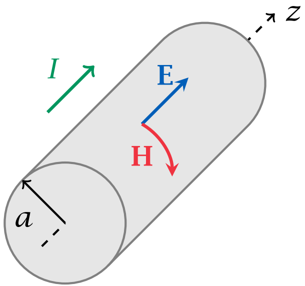
Example problem 4.4: Poynting vector for dc current

The Poynting vector is well defined also for static fields. Determine the Poynting vector in along straight conducting wire that carries a direct current I. The wire has radius a and conductivity σ. How is the result related to transmitted power and/or power loss in the wire?

Solution: Let the axis of the wire be the z axis with current flowing in the $+z$ direction. The dc current is uniformly distributed and we get the fields

$$\mathbf{E} = \mathbf{a}_z \frac{J}{\sigma} = \mathbf{a}_z \frac{I}{\pi a^2 \sigma},$$

$$\mathbf{H} = \mathbf{a}_\phi \frac{(r^2/a^2)I}{2\pi r} = \mathbf{a}_\phi \frac{rI}{2\pi a^2}.$$



The Poynting vector in the wire ($0 \le r \le a$) is

$$\mathscr{P} = \mathbf{E} \times \mathbf{H} = -\mathbf{a}_r \frac{rI^2}{2\pi^2 a^4 \sigma}.$$

This power flow radially inwards correspond to I^2R power loss in the wire.

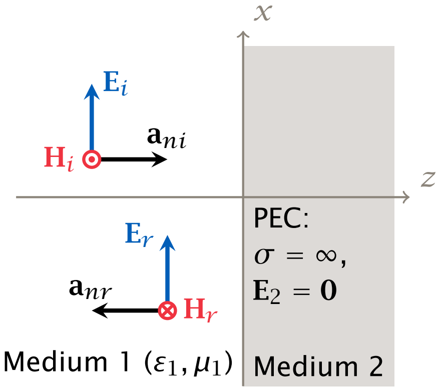
Next week

This is the end of lecture week 4. We'll continue with plane wave reflection and transmission (or refraction) at planar interfaces next week, by combining several plane waves to fulfill the boundary conditions at the interfaces.

Normal incidence at PEC boundary: standing waves

Cheng 8-6

Normal incidence at PEC boundary



$\mathbf{E} \times \mathbf{H} \upuparrows \mathbf{a}_n$ for any plane wave

In the ordinary lossless medium 1, we have an **incident wave (i)** and a **reflected wave (r)**:

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r = \mathbf{a}_x E_{i0} e^{-j\beta_1 z} + \mathbf{a}_x E_{r0} e^{+j\beta_1 z}$$

At $z = 0$ we have a PEC boundary:

$$\mathbf{a}_z \times \mathbf{E}_1 \Big|_{z=0} = \mathbf{0} \quad \Rightarrow \quad E_{r0} = -E_{i0}$$

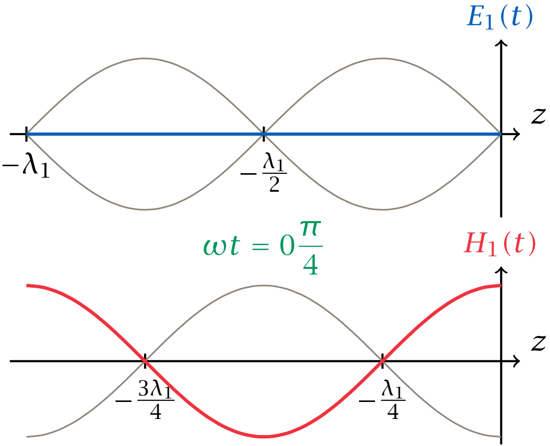
$$\Rightarrow \quad \mathbf{E}_1 = \mathbf{a}_x E_{i0} \left(e^{-j\beta_1 z} - e^{+j\beta_1 z} \right) = -\mathbf{a}_x j 2 E_{i0} \sin(\beta_1 z)$$

and since $\mathbf{H}_r \upuparrows \mathbf{H}_i$

$$\mathbf{H}_1 = \mathbf{a}_y 2 \frac{E_{i0}}{\eta_1} \cos(\beta_1 z)$$

Standing waves

The average power flow must be zero, since the incident and reflected waves transport equal power in opposite directions. It is also easy to see that $\mathscr{P}_{av} = \frac{1}{2}\mathscr{R}e\left[\mathbf{E}_1 \times \mathbf{H}_1^*\right] = \mathbf{0}$.



The total EM field in medium 1 is a **standing wave**

$$\mathbf{E}_1(t) = \mathbf{a}_x 2E_{i0}\sin(\beta_1 z)\sin(\omega t)$$

$$\mathbf{H}_1(t) = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1}\cos(\beta_1 z)\cos(\omega t)$$

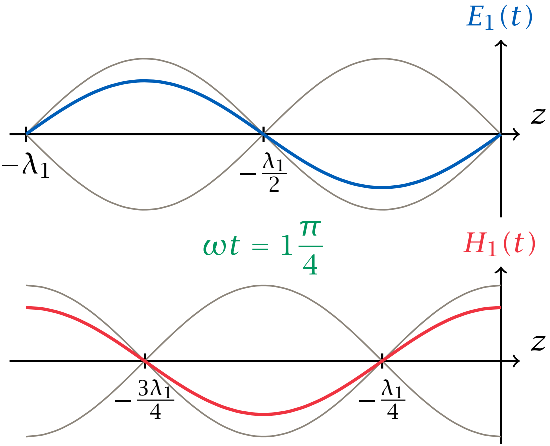
where zeros of $\mathbf{E}_1$ occurs at maximas of $\mathbf{H}_1$ and vice versa.

The wavelength in medium 1 is

$$\lambda_1 = \frac{2\pi}{\beta_1}, \quad \beta_1 = k_1 = \omega\sqrt{\mu_1\varepsilon_1}$$

Standing waves

The average power flow must be zero, since the incident and reflected waves transport equal power in opposite directions. It is also easy to see that
$\mathcal{P}_{\text{av}} = \frac{1}{2}\mathcal{Re}\left[\mathbf{E}_1 \times \mathbf{H}_1^*\right] = \mathbf{0}$.



The total EM field in medium 1 is a **standing wave**

$$\mathbf{E}_1(t) = \mathbf{a}_x 2E_{i0}\sin(\beta_1 z)\sin(\omega t)$$

$$\mathbf{H}_1(t) = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1}\cos(\beta_1 z)\cos(\omega t)$$

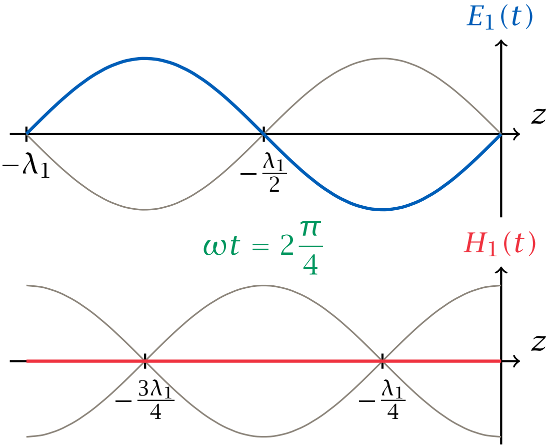
where zeros of $\mathbf{E}_1$ occurs at maximas of $\mathbf{H}_1$ and vice versa.

The wavelength in medium 1 is

$$\lambda_1 = \frac{2\pi}{\beta_1}, \quad \beta_1 = k_1 = \omega\sqrt{\mu_1\varepsilon_1}$$

Standing waves

The average power flow must be zero, since the incident and reflected waves transport equal power in opposite directions. It is also easy to see that $\mathscr{P}_{\mathrm{av}} = \frac{1}{2}\mathscr{Re}\left[\mathbf{E}_1 \times \mathbf{H}_1^*\right] = \mathbf{0}$.



The total EM field in medium 1 is a **standing wave**

$$\mathbf{E}_1(t) = \mathbf{a}_x 2E_{i0}\sin(\beta_1 z)\sin(\omega t)$$

$$\mathbf{H}_1(t) = \mathbf{a}_y 2\frac{E_{i0}}{\eta_1}\cos(\beta_1 z)\cos(\omega t)$$

where zeros of $\mathbf{E}_1$ occurs at maximas of $\mathbf{H}_1$ and vice versa.

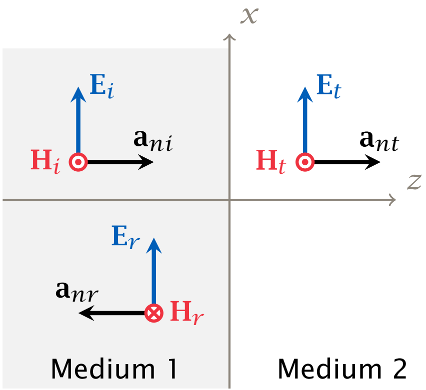
The wavelength in medium 1 is

$$\lambda_1 = \frac{2\pi}{\beta_1}, \quad \beta_1 = k_1 = \omega\sqrt{\mu_1\varepsilon_1}$$

Normal incidence at dielectric interface: reflection and transmission

Cheng 8-8

Normal incidence at dielectric interface



The same reference direction is again chosen for the E-fields.

In medium 1, we still have

$$\mathbf{E}_1 = \mathbf{E}_i + \mathbf{E}_r = \mathbf{a}_x E_{i0} e^{-j\beta_1 z} + \mathbf{a}_x E_{r0} e^{+j\beta_1 z}$$

and now we have a **transmitted wave (t)** in medium 2

$$\mathbf{E}_2 = \mathbf{E}_t = \mathbf{a}_x E_{t0} e^{-j\beta_2 z}$$

Tangential **E** and **H** are continuous at $z = 0$

$$\Rightarrow \begin{cases} E_{i0} + E_{r0} = E_{t0} \\ H_{i0} - H_{ir} = H_{t0} \end{cases}$$

Reflection and transmission coefficients

Expressing the two conditions as

$$E_{i0} + E_{r0} = E_{t0}, \qquad \frac{E_{i0}}{\eta_1} - \frac{E_{r0}}{\eta_1} = \frac{E_{t0}}{\eta_2}$$

and eliminating either E_{t0} or E_{r0}, we get the **reflection and transmission coefficients**

$$\boxed{\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \qquad \tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1} = 1 + \Gamma} \qquad \eta_i = \sqrt{\frac{\mu_i}{\varepsilon_i}}$$

This holds also for lossy media, but then the complex intrinsic impedances yield complex coefficients (= phase shift at reflection or transmission).

If medium 2 is PEC, $\eta_2 = 0$ and $\Gamma = -1$. Otherwise $|\Gamma| < 1$.

Example problem 5.1: Reflected and transmitted power

Since $\tau = 1 + \Gamma$ and Γ can be positive, the transmitted electric field may have larger amplitude than the incident one, but what does this mean for the transmitted power?

Calculate the average power density of the reflected and transmitted waves, when $\varepsilon_1 = 4\varepsilon_0$, $\varepsilon_2 = \varepsilon_0$, and the average power density of the incident plane wave is $99\,\mathrm{W/m^2}$. Both media are lossless dielectrics.

Solution

The reflection and transmission coefficients are

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\eta_0 - \eta_0/2}{\eta_0 + \eta_0/2} = \frac{1}{3}, \qquad \tau = 1 + \Gamma = \frac{4}{3}, \qquad \eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$$

The average power densities are

$$\mathcal{P}^i_{\text{av}} = \mathbf{a}_z \frac{E_{i0}^2}{2\eta_1} = \mathbf{a}_z \frac{E_{i0}^2}{\eta_0}, \qquad \mathcal{P}^r_{\text{av}} = -\mathbf{a}_z \frac{(\Gamma E_{i0})^2}{2\eta_1} = -\mathbf{a}_z \frac{1}{9} \frac{E_{i0}^2}{\eta_0}$$

$$\mathcal{P}^t_{\text{av}} = \mathbf{a}_z \frac{(\tau E_{i0})^2}{2\eta_2} = \mathbf{a}_z \frac{16}{9} \frac{E_{i0}^2}{2\eta_0} = \mathbf{a}_z \frac{8}{9} \frac{E_{i0}^2}{\eta_0}$$

So the reflected power density is $\frac{1}{9} \times 99\,\mathrm{W/m^2} = 11\,\mathrm{W/m^2}$ and the transmitted power density is $\frac{8}{9} \times 99\,\mathrm{W/m^2} = 88\,\mathrm{W/m^2}$ and the power balance is perfectly fine.

(Actually, the power reflection coefficient is $|\Gamma|^2$ and the power transmission coefficient is $1 - |\Gamma|^2$ for any two media and normal incidence.)

Total field in medium 1

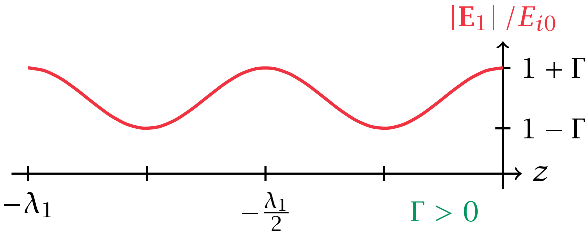
If medium 1 and 2 are lossless, $\eta_1, \eta_2, \Gamma, \tau \in \mathbb{R}$. For the total field in medium 1, we then have

$$\begin{aligned}\mathbf{E}_1 &= \mathbf{E}_i + \mathbf{E}_r = \mathbf{a}_x E_{i0}\left(e^{-j\beta_1 z} + \Gamma e^{+j\beta_1 z}\right) = \mathbf{a}_x E_{i0} e^{-j\beta_1 z}\left(1 + \Gamma e^{j2\beta_1 z}\right) \\ &= \mathbf{a}_x E_{i0}\left[(1+\Gamma)e^{-j\beta_1 z} + \Gamma\left(e^{+j\beta_1 z} - e^{-j\beta_1 z}\right)\right] \\ &= \mathbf{a}_x E_{i0}\left[\tau e^{-j\beta_1 z} + j2\Gamma\sin(\beta_1 z)\right] \\ &= \textbf{propagating wave + standing wave}\end{aligned}$$

The amplitude of $\mathbf{E}_1$ is between

$$E_{i0}(1+\Gamma), \quad z = -\frac{n\lambda_1}{2}, \; n = 0, 1, 2\ldots$$

$$E_{i0}(1-\Gamma), \quad z = -\frac{(2n+1)\lambda_1}{4}$$



Standing-wave ratio (SWR)

The ratio of the max and min amplitudes of the electric field is the **standing-wave ratio (SWR)**

$$\boxed{S = \frac{|E|_{\text{max}}}{|E|_{\text{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}}$$

Note that $\Gamma \in \mathbb{C}$ in general. On the previous slide we assumed lossless media.

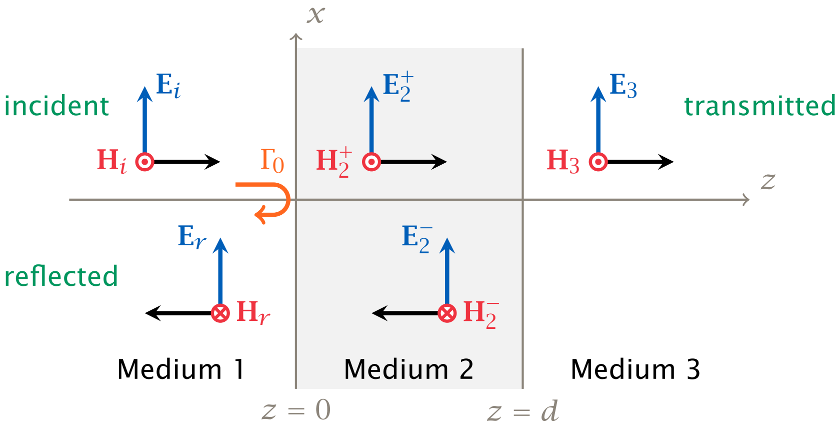
Since $S \geq 1$ without upper limit, SWR is often given in decibels

$$S_{\text{dB}} = 10 \log_{10}(S^2) = 20 \log_{10}(S)$$

Multiple dielectric layers

Cheng 8-9

Normal incidence at two dielectric interfaces



Five plane waves are needed to construct the general solution. The total fields must satisfy the interface conditions at $z = 0$ and $z = d$. Let's use the previous result for a single interface to solve the reflection coefficient Γ_0.

Wave impedance of the total field

The **wave impedance of the total field** is the ratio of the total electric field and the total magnetic field. For the total EM fields in medium 1 in the previous single interface situation, we have

$$Z_1(z) = \frac{E_{1x}(z)}{H_{1y}(z)} = \frac{E_{i0}\left(e^{-j\beta_1 z} + \Gamma e^{+j\beta_1 z}\right)}{\frac{E_{i0}}{\eta_1}\left(e^{-j\beta_1 z} - \Gamma e^{+j\beta_1 z}\right)} = \eta_1 \frac{e^{-j\beta_1 z} + \Gamma e^{+j\beta_1 z}}{e^{-j\beta_1 z} - \Gamma e^{+j\beta_1 z}}.$$

Substituting $\Gamma = (\eta_2 - \eta_1)/(\eta_2 + \eta_1)$ and $z = -\ell$, we get

$$Z_1(-\ell) = \cdots = \eta_1 \frac{\eta_2 \cos(\beta_1 \ell) + j\eta_1 \sin(\beta_1 \ell)}{\eta_1 \cos(\beta_1 \ell) + j\eta_2 \sin(\beta_1 \ell)}.$$

Now, we get the wave impedance for the total field at $z = 0$ in medium 2 in the double interface situation by replacing $\ell \rightarrow d$ and adding 1 to the subscripts.

Reflection coefficient at the first interface

The wave impedance for the total field is

$$Z_2(0) = \eta_2 \frac{\eta_3 \cos(\beta_2 d) + j\eta_2 \sin(\beta_2 d)}{\eta_2 \cos(\beta_2 d) + j\eta_3 \sin(\beta_2 d)}.$$

The reflection coefficient Γ_0 depend on the total fields at the interface $z = 0$. We get the correct $\mathbf{E}_2(0)$ and $\mathbf{H}_2(0)$ (or at least the correct ratio), if we replace medium 2 and 3 with a single homogeneous medium with intrinsic impedance $Z_2(0)$. Hence, the reflection coefficient is

$$\Gamma_0 = \frac{E_{r0}}{E_{i0}} = \frac{Z_2(0) - \eta_1}{Z_2(0) + \eta_1}.$$

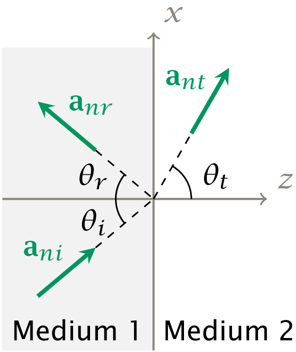
This process could be continued to solve the reflection from 3 or more interfaces.

Oblique incidence at dielectric (or PEC) interface

Cheng 8-10 (and 8-7)

Oblique reflection and transmission (refraction)

When medium 1 and 2 are simple lossless media



θ_i = angle of incidence
θ_r = angle of reflection
θ_t = angle of transmission
(or refraction)

General plane wave in lossless medium

$$\mathbf{E} = \mathbf{E}_0\, e^{-j\beta \mathbf{a}_n \cdot \mathbf{R}}, \quad \beta = k = \omega\sqrt{\mu\varepsilon}$$

Incident, reflected, and transmitted wave

$$\mathbf{E}_i = \mathbf{E}_{i0}\, e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$
$$\mathbf{E}_r = \mathbf{E}_{r0}\, e^{-j\beta_1 (x\sin\theta_r - z\cos\theta_r)}$$
$$\mathbf{E}_t = \mathbf{E}_{t0}\, e^{-j\beta_2 (x\sin\theta_t + z\cos\theta_t)}$$

Tangential $\mathbf{E}$ is continuous at the interface

$$\Leftrightarrow \quad \mathbf{a}_z \times (\mathbf{E}_i + \mathbf{E}_r) = \mathbf{a}_z \times \mathbf{E}_t,$$

for all x, when $z = 0$.

Interface conditions ⇒ Snell's laws

At the boundary, for all x

$$\mathbf{a}_z \times \mathbf{E}_i \, e^{-j\beta_1 x \sin\theta_i} + \mathbf{a}_z \times \mathbf{E}_r \, e^{-j\beta_1 x \sin\theta_r} = \mathbf{a}_z \times \mathbf{E}_t \, e^{-j\beta_2 x \sin\theta_t}.$$

This is possible if and only if the waves are in-phase

$$\beta_1 \sin\theta_i = \beta_1 \sin\theta_r = \beta_2 \sin\theta_t$$

This gives **Snell's law of reflection and refraction**

$$\boxed{\theta_i = \theta_r, \qquad n_1 \sin\theta_i = n_2 \sin\theta_t}$$

where the **index of refraction** of medium 1 is

$$n_1 = \frac{\beta}{k_0} = \sqrt{\mu_{r1}\varepsilon_{r1}} = \frac{c}{u_p}$$ (and similarly for medium 2)

Note that $n = \sqrt{\mu_r \varepsilon_r} \neq \eta = \sqrt{\mu/\varepsilon}$

Total reflection

If $n_1 > n_2$, there is incident angle $\theta_i = \theta_c$ that gives $\theta_t = \pi/2$:

$$n_1 \sin \theta_c = n_2 \sin \frac{\pi}{2} \quad \Leftrightarrow \quad \boxed{\sin \theta_c = \frac{n_2}{n_1}}$$

If the incident angle θ_i is larger than the **critical angle θ_c**, there is **total reflection** (100% of the incident power is reflected).

However, the fields in medium 2 is not zero. When $\theta_i > \theta_c$, Snell's law of refraction gives a complex angle of refraction that corresponds to a nonuniform plane wave in medium 2.

Evanescent wave or surface wave in medium 2

Let us study the example $n_1 = \sqrt{2}$, $n_2 = 1$, and $\theta_i = \pi/3 > \theta_c = \pi/4$. Snells law gives $\theta_t \approx \pi/2 \pm j\,0.658$, which is difficult to interpret geometrically. On the other hand,

$$\sin\theta_t = \frac{n_1}{n_2}\sin\theta_i = \sqrt{\frac{3}{2}}, \quad \cos\theta_t = \pm\sqrt{1-\sin^2\theta_t} = \pm j\frac{1}{\sqrt{2}}.$$

If we choose the minus sign,

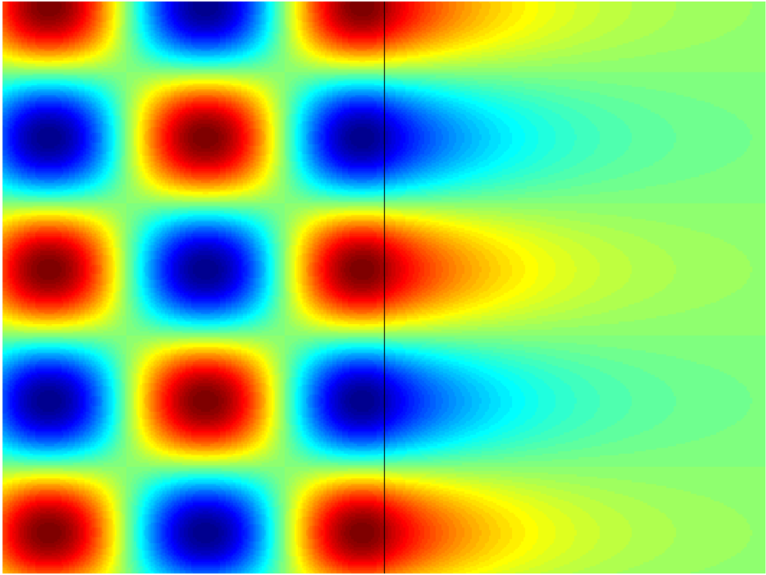
$$\mathbf{E}_t = \mathbf{E}_{t0}\, e^{-j\beta_2(x\sin\theta_t + z\cos\theta_t)} = \mathbf{E}_{t_0}\, e^{-j\left(\beta_2\sqrt{3/2}\right)x}\, e^{-(\beta_2/\sqrt{2})z}.$$

This nonuniform plane wave is an **evanescent wave** that decays exponentially in the $+z$ direction while propagating in the $+x$ direction along the interface. It is a **surface wave** since it is bound to the surface.

Instead of the complex angle p of transmission, we can express the wave using the complex wavevector $\mathbf{k}_t = \beta_2\sqrt{3/2}\,\mathbf{a}_x - j(\beta_2/\sqrt{2})\,\mathbf{a}_z$.

Time-dependent y-polarized electric field at total reflection

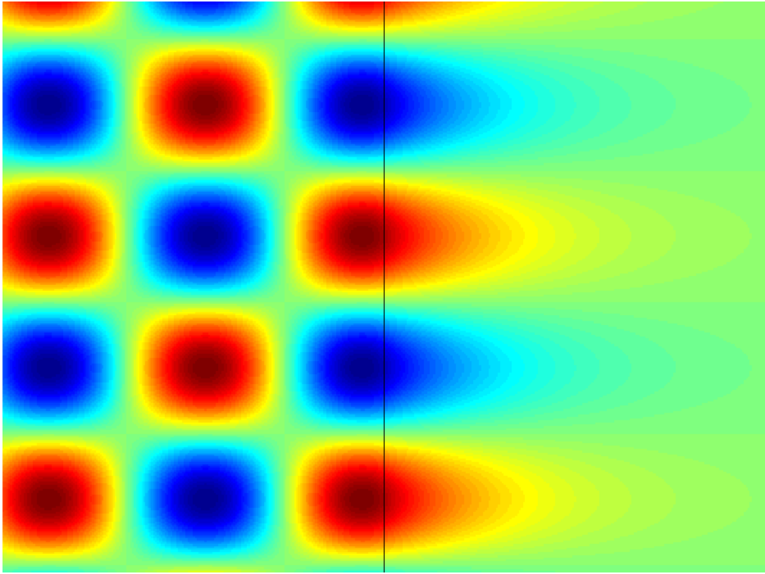
Plot of $E_y(t)$ in zx-plane, when $\varepsilon_1 = 2\varepsilon_0, \quad \varepsilon_2 = \varepsilon_0, \quad \theta_i = 50^\circ > \theta_c = 45^\circ$



$\omega t = 0$

Time-dependent y-polarized electric field at total reflection

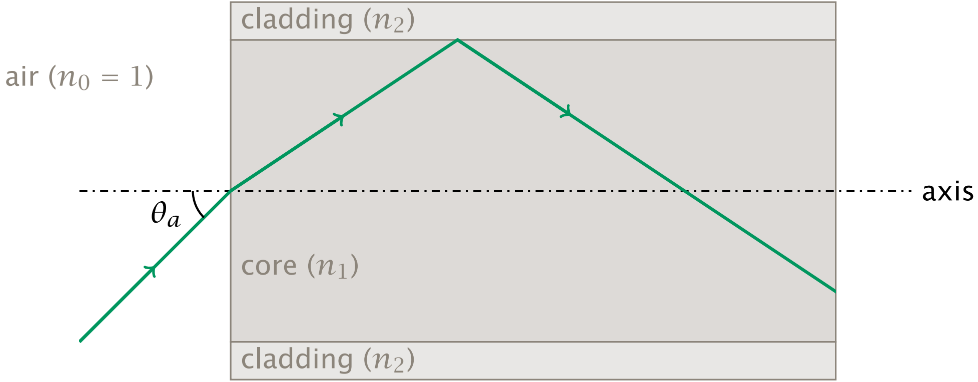
Plot of $E_y(t)$ in zx-plane, when $\varepsilon_1 = 2\varepsilon_0$, $\varepsilon_2 = \varepsilon_0$, $\theta_i = 50^\circ > \theta_c = 45^\circ$



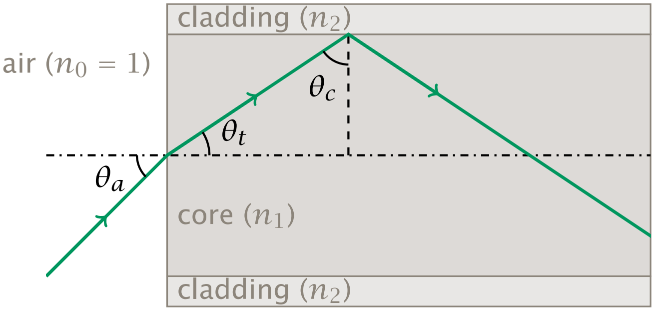
$\omega t = \pi/4$

Example problem 5.2: Optical fiber

A simple model for an optical fiber would be a homogeneous lossless dielectric core with refractive index n_1 surrounded by a cladding with $n_2 < n_1$. Express the maximum angle of incidence (or acceptance angle) θ_a, when the rays are trapped inside the core by total internal reflection.



Solution



Total reflection:

$$\sin\theta_c = \frac{n_2}{n_1}$$

Snell's law:

$$\sin\theta_a = n_1 \sin\theta_t$$

Trigonometry:

$$\sin\theta_t = \cos\theta_c = \sqrt{1 - \sin^2\theta_c}$$

$$\Rightarrow \quad \sin\theta_a = n_1 \sin\theta_t = n_1\sqrt{1 - (n_2/n_1)^2} = \sqrt{n_1^2 - n_2^2}$$

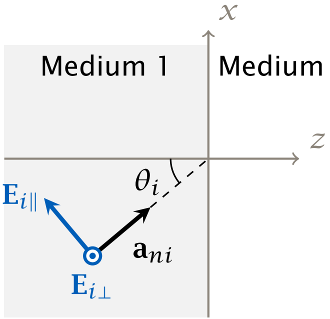
Polarization?

Snell's law does not depend on polarization, and so the critical angle is also independent of polarization.

However, the reflection and transmission coefficients are polarization-dependent.

Plane of incidence and polarizations

The **plane of incidence** is spanned by $\mathbf{a}_{ni}$ and the normal vector of the interface.

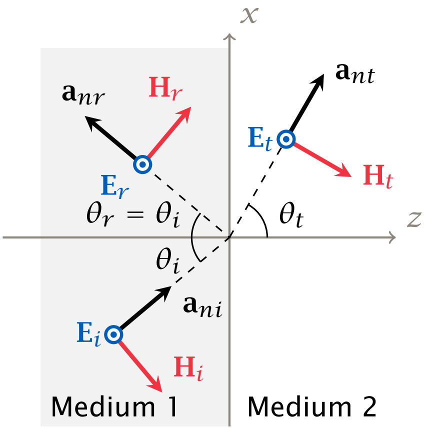


With respect to this (xz) plane, we have

- **Perpendicular polarization** $\mathbf{E}_{i\perp}$ (horizontal polarization, E-polarization, or TE polarization)
- **Parallel polarization** $\mathbf{E}_{i\parallel}$ (vertical polarization, H-polarization, or TM polarization)

- Any incident plane wave can be decomposed in these two polarizations.
- These polarizations are preserved in reflection and refraction.

Perpendicular polarization



At the boundary:

$$\mathbf{E}_i + \mathbf{E}_r = \mathbf{E}_t$$

$$\mathbf{a}_x \cdot (\mathbf{H}_i + \mathbf{H}_r) = \mathbf{a}_x \cdot \mathbf{H}_t$$

Incident wave:

$$\mathbf{E}_i = \mathbf{a}_y E_{i0}\, e^{-j\beta_1 \mathbf{a}_{ni} \cdot \mathbf{R}}$$

$$\mathbf{H}_i = (-\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i)\, \frac{E_{i0}}{\eta_1}\, e^{-j\beta_1 \mathbf{a}_{ni} \cdot \mathbf{R}}$$

Reflected wave:

$$\mathbf{E}_r = \mathbf{a}_y E_{r0}\, e^{-j\beta_1 \mathbf{a}_{nr} \cdot \mathbf{R}}$$

$$\mathbf{H}_r = (\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i)\, \frac{E_{r0}}{\eta_1}\, e^{-j\beta_1 \mathbf{a}_{nr} \cdot \mathbf{R}}$$

Transmitted wave:

$$\mathbf{E}_t = \mathbf{a}_y E_{t0}\, e^{-j\beta_2 \mathbf{a}_{nt} \cdot \mathbf{R}}$$

$$\mathbf{H}_t = (-\mathbf{a}_x \cos\theta_t + \mathbf{a}_z \sin\theta_t)\, \frac{E_{t0}}{\eta_2}\, e^{-j\beta_2 \mathbf{a}_{nt} \cdot \mathbf{R}}$$

Snell: the phase factor is the same when $z = 0$.

Reflection and transmission coefficients

Continuity of E_y and H_x at the boundary $z = 0$ gives the conditions

$$E_{i0} + E_{r0} = E_{t0}, \qquad -\cos\theta_i \frac{E_{i0}}{\eta_1} + \cos\theta_i \frac{E_{r0}}{\eta_1} = -\cos\theta_t \frac{E_{t0}}{\eta_2}$$

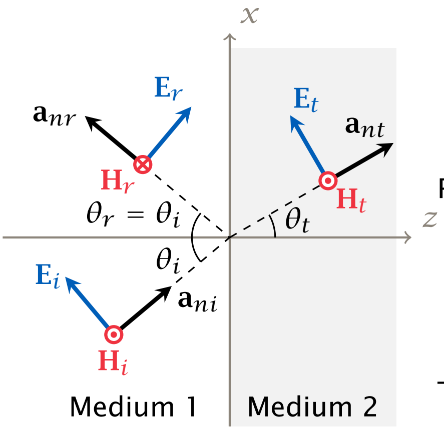
From these we can solve the **reflection and transmission coefficients for perpendicular polarization**

$$\boxed{\Gamma_\perp = \frac{E_{0r}}{E_{0i}} = \frac{\eta_2/\cos\theta_t - \eta_1/\cos\theta_i}{\eta_2/\cos\theta_t + \eta_1/\cos\theta_i}, \qquad \tau_\perp = \frac{E_{t0}}{E_{i0}} = 1 + \Gamma_\perp}$$

These formulas are also known as Fresnel's equations.

Note that $\eta/\cos\theta = E_y/H_x$.

Parallel polarization



At the boundary:

$$\mathbf{a}_x \cdot (\mathbf{E}_i + \mathbf{E}_r) = \mathbf{a}_x \cdot \mathbf{E}_t$$
$$\mathbf{H}_i + \mathbf{H}_r = \mathbf{H}_t$$

Incident wave:

$$\mathbf{E}_i = (\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)\, E_{i0}\, e^{-j\beta_1 \mathbf{a}_{ni}\cdot\mathbf{R}}$$
$$\mathbf{H}_i = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 \mathbf{a}_{ni}\cdot\mathbf{R}}$$

Reflected wave:

$$\mathbf{E}_r = (\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i)\, E_{r0}\, e^{-j\beta_1 \mathbf{a}_{nr}\cdot\mathbf{R}}$$
$$\mathbf{H}_r = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{-j\beta_1 \mathbf{a}_{nr}\cdot\mathbf{R}}$$

Transmitted wave:

$$\mathbf{E}_t = (\mathbf{a}_x \cos\theta_t - \mathbf{a}_z \sin\theta_t)\, E_{t0}\, e^{-j\beta_2 \mathbf{a}_{nt}\cdot\mathbf{R}}$$
$$\mathbf{H}_t = \mathbf{a}_y \frac{E_{t0}}{\eta_1} e^{-j\beta_2 \mathbf{a}_{nt}\cdot\mathbf{R}}$$

Snell: the phase factor is the same when $z = 0$.

Reflection and transmission coefficients

Continuity of E_x and H_y at the boundary $z = 0$ gives the conditions

$$E_{i0} \cos\theta_i + E_{r0} \cos\theta_i = E_{t0} \cos\theta_t, \qquad \frac{E_{i0}}{\eta_1} - \frac{E_{r0}}{\eta_1} = \frac{E_{t0}}{\eta_2}$$

From these we can solve the **reflection and transmission coefficients for parallel polarization**

$$\boxed{\Gamma_\parallel = \frac{E_{0r}}{E_{0i}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}, \qquad \tau_\parallel = \frac{E_{t0}}{E_{i0}} = (1 + \Gamma_\parallel)\frac{\cos\theta_i}{\cos\theta_t}}$$

These formulas are also known as Fresnel's equations.

Note that $\eta \cos\theta = E_x / H_y$.

Special cases

Perfect electric conductor

If medium 2 is PEC, we have $\eta_2 = 0$ and

$$\Gamma_\perp = \Gamma_\parallel = -1$$

and $\tau_\perp = \tau_\parallel = 0$, although θ_t is undefined.

Normal incidence

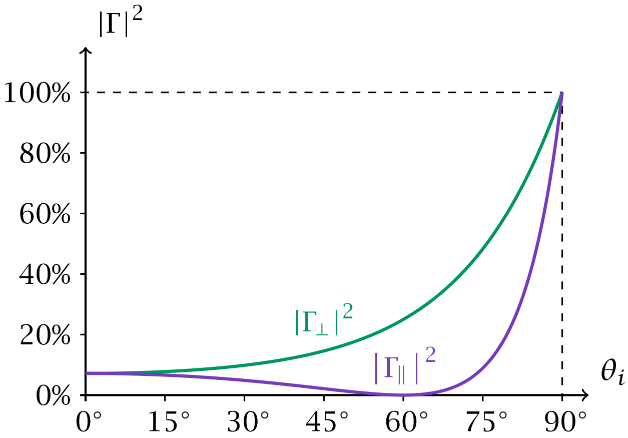
If $\theta_i = 0$, Snell's laws give $\theta_r = \theta_t = 0$ and

$$\Gamma_\perp = \Gamma_\parallel = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}, \qquad \tau_\perp = \tau_\parallel = \frac{2\eta_2}{\eta_2 + \eta_1},$$

as expected.

Note that the reference directions for $\mathbf{E}$ and $\mathbf{H}$ are in all cases selected so that the direction of $\mathbf{E}$ is the same for all waves when $\theta_i \to 0$.

Example: Light reflection from a glass surface



With $\varepsilon_1 = \varepsilon_0$ and $\varepsilon_2 = 3\varepsilon_0$, we get

- 7.2% power reflection at normal incidence ($\Gamma_\perp = \Gamma_\parallel = -0.269$)
- $\Gamma_\perp \to -1$ and $\Gamma_\parallel \to +1$ when $\theta_i \to 90°$
- $\Gamma_\parallel = 0$ at $\theta_i = 60°$

For any nonmagnetic interface ($\mu_1 = \mu_2 = \mu_0$) the reflection of perpendicular polarization is stronger: $|\Gamma_\perp|^2 > |\Gamma_\parallel|^2$ when $0 < \theta_i < 90°$.

The Brewster angle of no reflection

The **Brewster angle** $\theta_{B\parallel}$ (or $\theta_{B\perp}$) is the incident angle where $\Gamma_\parallel = 0$ (or $\Gamma_\perp = 0$).

For parallel polarization and lossless dielectrics, we get

$$\Gamma_\parallel = 0 \quad \Rightarrow \quad \eta_2 \cos\theta_t - \eta_1 \cos\theta_{B\parallel} = 0 \quad \Leftrightarrow \quad \sqrt{\varepsilon_1}\cos\theta_t = \sqrt{\varepsilon_2}\cos\theta_{B\parallel}$$

combining this with Snell's law

$$n_1 \sin\theta_{B\parallel} = n_2 \sin\theta_t \quad \Rightarrow \quad \cos^2\theta_t = 1 - \sin^2\theta_t = 1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_{B\parallel}$$

we get

$$\varepsilon_1\left(1 - \frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_{B\parallel}\right) = \varepsilon_2\left(1 - \sin^2\theta_{B\parallel}\right) \quad \Rightarrow \quad \boxed{\sin\theta_{B\parallel} = \frac{1}{\sqrt{1 + \varepsilon_1/\varepsilon_2}}}$$

This has a real solution ($0 < \theta_{B\parallel} < \pi/2$) for any permittivity ratio $\varepsilon_1/\varepsilon_2 > 0$.

Example problem 5.3: Brewster angle

For ordinary media with parameters ε_1, μ_1 and ε_2, μ_2, derive the Brewster angle for perpendicular polarization $\theta_{B\perp}$.

Solution:

$$\Gamma_\perp = 0 \quad \Rightarrow \quad \frac{\sqrt{\mu_2/\varepsilon_2}}{\cos\theta_t} - \frac{\sqrt{\mu_1/\varepsilon_1}}{\cos\theta_{B\perp}} = 0 \quad \Leftrightarrow \quad \frac{\mu_2}{\varepsilon_2}\cos^2\theta_{B\perp} = \frac{\mu_1}{\varepsilon_1}\cos^2\theta_t$$

Combining this with Snell's law

$$\sqrt{\mu_1\varepsilon_1}\sin\theta_{B\perp} = \sqrt{\mu_2\varepsilon_2}\sin\theta_t \quad \Rightarrow \quad \cos^2\theta_t = 1 - \sin^2\theta_t = 1 - \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_{B\perp}$$

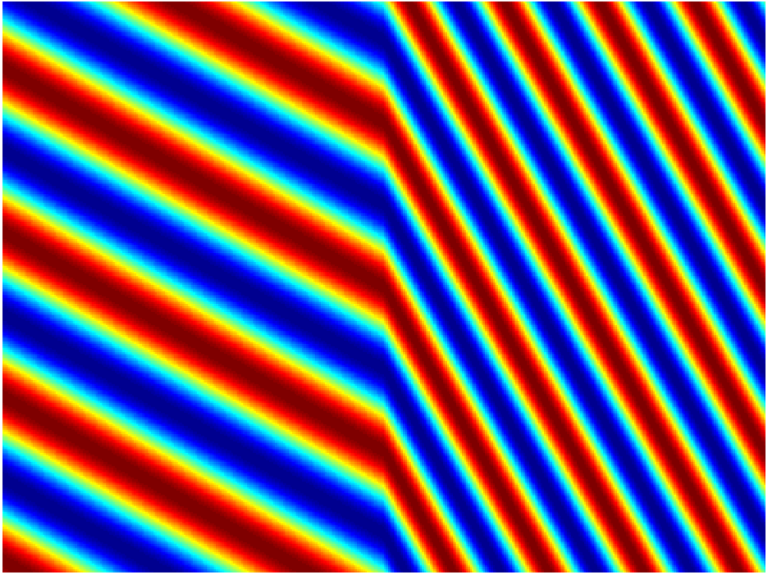
gives

$$\frac{\mu_2}{\varepsilon_2}\left(1 - \sin^2\theta_{B\perp}\right) = \frac{\mu_1}{\varepsilon_1}\left(1 - \frac{\mu_1\varepsilon_1}{\mu_2\varepsilon_2}\sin^2\theta_{B\perp}\right) \quad \Rightarrow \quad \boxed{\sin\theta_{B\perp} = \sqrt{\frac{1 - \frac{\mu_1\varepsilon_2}{\mu_2\varepsilon_1}}{1 - \left(\frac{\mu_1}{\mu_2}\right)^2}}}$$

This has a real solution only when $\mu_1 \neq \mu_2$.

Parallel polarization, $H_y(t)$ in zx plane

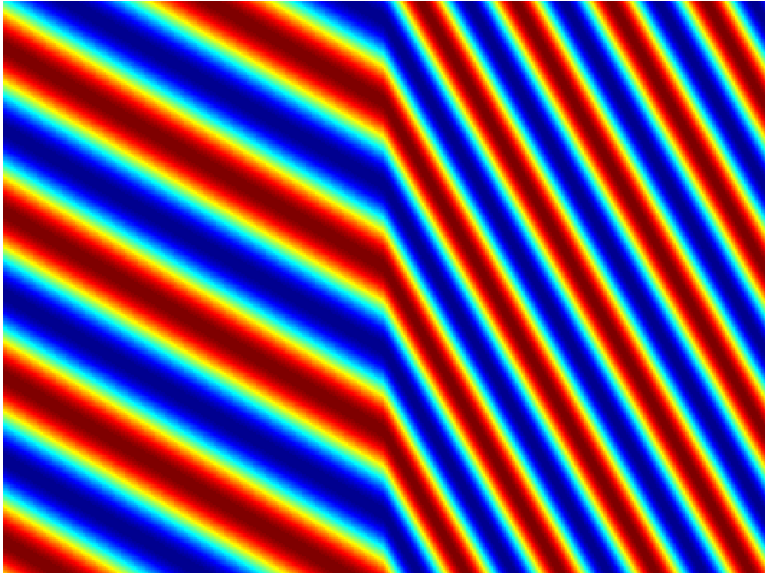
$\varepsilon_1 = \varepsilon_0, \quad \varepsilon_2 = 3\varepsilon_0, \quad \theta_i = 60^\circ, \quad \theta_t = 30^\circ, \quad \Gamma_\parallel = 0, \quad \tau_\parallel = 1/\sqrt{3}$



$\omega t = 0$

Parallel polarization, $H_y(t)$ in zx plane

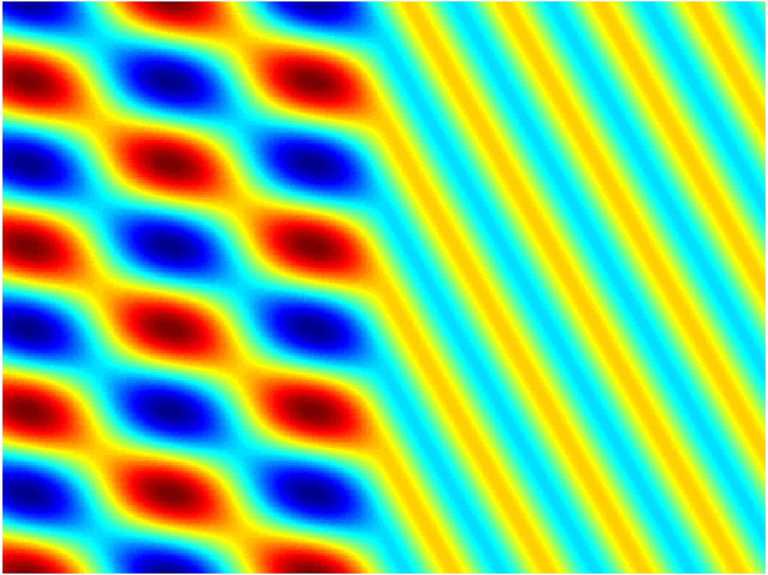
$\varepsilon_1 = \varepsilon_0, \quad \varepsilon_2 = 3\varepsilon_0, \quad \theta_i = 60^\circ, \quad \theta_t = 30^\circ, \quad \Gamma_\parallel = 0, \quad \tau_\parallel = 1/\sqrt{3}$



$\omega t = \pi/4$

Perpendicular polarization, $E_y(t)$ in zx plane

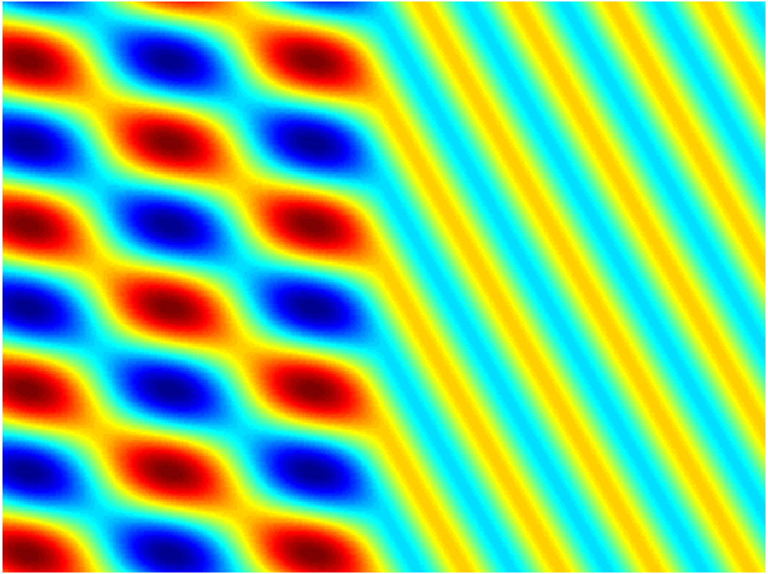
$\varepsilon_1 = \varepsilon_0, \quad \varepsilon_2 = 3\varepsilon_0, \quad \theta_i = 60^\circ, \quad \theta_t = 30^\circ, \quad \Gamma_\perp = -1/2, \quad \tau_\perp = 1/2$



$\omega t = 0$

Perpendicular polarization, $E_y(t)$ in zx plane

$\varepsilon_1 = \varepsilon_0, \quad \varepsilon_2 = 3\varepsilon_0, \quad \theta_i = 60^\circ, \quad \theta_t = 30^\circ, \quad \Gamma_\perp = -1/2, \quad \tau_\perp = 1/2$



$\omega t = \pi/4$

Outlook

This is the end of lecture week 5. Prof. Taylor will continue next week:

Week 6 Theory and applications of transmission lines (Taylor)

Week 7 Midterm 1 covering the content of weeks 1–5

Week 8 Theory and applications of transmission lines + ML (Taylor)

Week 9 Multilayer calculations (Taylor)

Week 10 Waveguides and cavity resonators (Taylor)

Week 11 Waveguides and cavity resonators (Taylor)

Week 12 Antennas and radiating systems (Wallén)

Week 13 Antennas and radiating systems (Wallén)

Week 14 Midterm 2 covering the content of weeks 6, 8–13

Chapters 9–11 in Cheng + some extra on ML