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Lecture 5: Plasma dynamics
Electrostatic waves

Today's menu

- From perturbations to oscillations to waves:
  - Plane waves, Fourier Transformation, and linearization
- Boltzmann relation
- Equation of state & degrees of freedom
- ES waves with no B-field
  - Electron plasma waves
  - Ion acoustic wave
- ES waves with $B \neq 0$:
  - Electrons: *upper hybrid frequency*
  - Ions: *lower hybrid frequency & electrostatic ion cyclotron wave*

Notational issues …

In these lectures we shall have our mouthful of frequencies and a notational mumbo-jumbo lurks behind the corner. We shall try our best to abide in the following conventions:

Larmor frequency:

capital $\Omega = qB/m$, separate for electrons and ions

This can also be called gyro or cyclotron frequency, depending on the application.

Plasma frequency: $\omega_p = \sqrt{e^2 n/\epsilon_0 m}$, also separately for e & i

Perturb a *cold* plasma

We have actually already perturbed a plasma in the 1st lecture

➔ We found that the plasma responded by oscillating around the equilibrium position with frequency called *the plasma frequency,*

$$\omega_p = \sqrt{en/4\pi\epsilon_0 m}.$$

But the analysis contained some implicit *assumptions…*

Let's now take into account that particles in plasma (actually *electrons*) have kinetic energy, parameterized by temperature, T.

Basic formalism:
perturbation theory and plane waves

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Perturbations vs oscillations

The basic assumption is that the perturbation is *small:*

For instance, the density: $n = n_0 + \delta n$, with $\frac{\delta n}{n_0} \ll 1$

Small perturbation around equilibrium … ring a bell… HO !!!

➔Solutions expected oscillatory, at least in the first approximation

➔ solutions sines and/or cosines

Or rather: $\delta n \propto exp\big(i(kx - \omega t)\big)$; derivatives become products!

$$\frac{\partial}{\partial t} \rightarrow -i\omega \quad \text{and} \quad \boldsymbol{\nabla} \rightarrow i\boldsymbol{k}$$

A justification for the *harmonic approximation*

- A euclidian 3D space spanned by any 3 vectors that are orthogonal to each other, e.g. $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$. These form a *complete base* for vectors, i.e., any vector can be expressed as a linear combination of them.
- Same in the functional space: ∃ functions that form a complete base for *functions*
- Periodic function f with period L: $f(x) = a_0 + \sum a_i \cos k_i x + b_i \sin k_i x$; $k_i = i\frac{2\pi}{L}$.
- Let $L \to \infty$, ➔ $\sum \to \int$: $f(x) = \int_{-\infty}^{\infty} g(k) e^{-ikx} dk$
- In plane... no, *plain* English:

Function $f(x,t)$ can be expressed as a linear combination of plane waves $e^{-(ikx - i\omega t)}$

Another look at the Fourier transformation

➔ A Fourier transformation can be viewed as a wave package of plane waves, each with a k-dependent amplitude $g(k)$:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int g(k) e^{i(kx-\omega t)} dk$$

➔ if we know the function, we can find how strongly each plane wave component contributes to it

$$g(k) = \frac{1}{\sqrt{2\pi}} \int f(x) e^{-i(kx-\omega t)} dx$$

➔ Since the harmonic plane waves are linearly independent, we can solve the equations independently for each k and 'sum' them up in the end !

Linearization of the equations

If the perturbation is small, we can *linearize* the equations, i.e., throw out all terms that are higher power in perturbed quantities

Notation: any physical quantity $f = f_0 + f_1$, with $\frac{f_1}{f_0} \ll 1$.

Let's practise with the continuity equation:

$$\frac{\partial(n_0 + n_1)}{\partial t} + \boldsymbol{\nabla} \cdot (n_0 + n_1)(\boldsymbol{v_0} + \boldsymbol{v_1}) = 0$$

Assume initially stationary ($\boldsymbol{v_0} = 0$) and homogeneous ($\boldsymbol{\nabla} n_0 = 0$) plasma & linearize:

$$\frac{\partial(n_1)}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{v_1} = 0$$

First application of linearization: *the Boltzmann relation*

Equation of motion for electron fluid : $mn\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}\right] = -en\boldsymbol{E} - \nabla p$

- Assume n_0=constant, $\boldsymbol{v_0} = 0$, T_e = constant (isothermal plasma)
- Set z-axis so that $\nabla p = \frac{\partial p}{\partial z}$

Linearize ➔ $\frac{\partial \boldsymbol{v}_1}{\partial t} = -e\boldsymbol{E}_1/m - \frac{T_e}{mn}\frac{\partial n}{\partial z}$

If we can neglect electron inertia ($m \to 0$) but keep $v < \infty$, the terms on the RHS have to balance: $e\frac{\partial \phi}{\partial z} = \frac{T_e}{n}\frac{\partial n}{\partial z}$; $E_1 = -\frac{\partial \phi}{\partial z}$

$\int dz$ ➔ e$\phi = T_e \log n + C$

➔ ***Boltzmann relation***: $n_e = n_0 e^{e\phi/T_e}$

Electrostatic waves in non-magnetized plasmas

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Pertubations, oscillations and waves

We actually perturbed the plasma already in the first lecture

➔ Plasma (electron) oscillations with $\omega = \omega_{pe} = \sqrt{en/4\pi\epsilon_0 m}$

There we assumed cold plasma, $T_e = 0$.

Let us now (re-)analyze using the full set of *fluid equations* what happens when we perturb the plasma

We will find that a perturbation can *propagate as a wave*.

Note: in neutral gas we have a single wave: the sound wave.

In plasmas we will find N waves with $N \gg 1$ … ☺

Definition of a wave

A wave is propagation with periodic motion characterized by

- Wavelength λ
- Wave number $k = \frac{2\pi}{\lambda}$
- Angular frequency ω
- Amplitude A

The *phase velocity*, $v_{ph} = \omega / k$ characterizes motion of wave crests

The *group velocity* $v_{gr} = \frac{d\omega}{dk}$ gives the speed at which the full wave package, i.e., information can propagate.

Why frequency is important

If the frequency at which the plasma responds to the perturbation is sufficiently high, the ions with large inertia cannot respond

➔ take ions as immobile positive charge background and analyze only electron dynamics.

Dynamics requires including electron *inertia*, which is the force counter-acting external perturbation.

But first, a closer look at the equation of state …

Equation of state and degrees of freedom

The equation of state was adopted to close the set of equations

$p = Cn^{\gamma}$, where $\gamma = \frac{N+2}{N}$, with N = # of degrees of freedom.

$$\Rightarrow \frac{\nabla p}{p} = \gamma \frac{\nabla n}{n}$$

Special case: isothermal plasma, $T = const. \rightarrow \nabla p = T\nabla n$

So for isothermal plasma $\gamma = 1$.

Generating electron plasma vibes

Hit the plasma w/ a hammer = 1D perturbation ➔ γ = 3.

Unperturbed plasma:

- n_0=const., $\boldsymbol{v_0}$ = 0, $\boldsymbol{E_0}$ = 0, $T_e = const$

Linearize EoM: $mn_0 \frac{\partial v_1}{\partial t} = -enE_1 - 3T_e \frac{\partial n_1}{\partial x}$

x-axis along perturbation & FT ➔ $-i\omega mn_0 v_1 = -en_0 E_1 - ik3T_e n_1$

Need equations to link v_1, E_1 and n_1…

Eliminating unknowns

Continuity equation: $\frac{\partial n}{\partial t} + \nabla \cdot (n\boldsymbol{v}) = 0$ ➔ $v_1 \leftrightarrow n_1$

Linearize & FT ➔ $-i\omega n_1 + n_0 ikv_1 = 0 \rightarrow n_1 = \frac{kn_0}{\omega} v_1$

How about E_1?

Now the density *is locally* disturbed ➔ $\nabla \cdot \boldsymbol{E} = \frac{e}{4\pi\epsilon_0}\left(n_0 - (n_0 + n_1)\right)$

Linearize & FT ➔ $E_1 = i\frac{en_1}{4\pi\epsilon_0 k}$

Electron plasma wave

Express now everything in terms of v_1 in the electron EoM

➔ $i\omega m n_0 v_1 = \left[e n_0 \frac{ie}{4\pi\epsilon_0 k} + 3ikT_e \right] \frac{n_0 k}{\omega} v_1$

➔ $\omega^2 = \frac{e^2 n_0}{4\pi\epsilon_0 m} + 3k^2 T_e / m = \omega_{pe}^2 + \frac{3}{2} k^2 v_{th}^2$, where $T = \frac{1}{2} m v_{th}^2$

This is the *dispersion relation* for the electron plasma waves: each wavelength has slightly different frequency, thus allowing information to propagate:

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 v_{th}^2$$

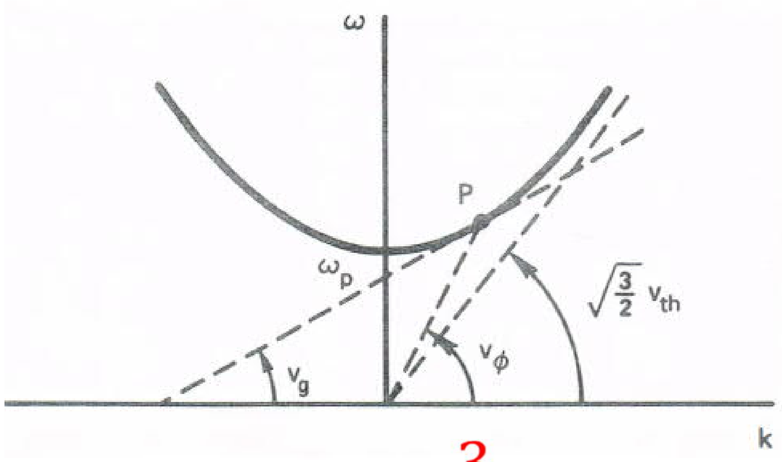
Physics of electron plasma wave

Changes in plasma quantities propagate at the speed

$$v_{gr} = \frac{d\omega}{dk} = \frac{3}{2}\frac{k}{\omega}v_{th}^2$$

How come we got plain plasma oscillations earlier?

When $T \to 0$, we get $\omega^2 = \omega_p^2$. Consistent with our assumptions! ☺



$$\omega^2 = \omega_p^2 + \frac{3}{2}k^2 v_{th}^2$$

What happens in longer time scales?

Even the ions will respond to the perturbation.

NOTE: here we cannot ignore the other component of the plasma, the electrons, since they will be heeding the ions like faithful dogs…

But first, the ions (same initial assumptions as before):

$$Mn\frac{\partial \boldsymbol{v}}{\partial t} = -en\boldsymbol{E} - \nabla p$$

Small perturbation, linearize & use $\nabla p = \gamma_i T \nabla n$

$$\Rightarrow Mn_0\frac{\partial v_1}{\partial t} = enE_1 - \gamma_i T_i \frac{\partial n_1}{\partial x}$$

How about the electrons at these times?

For long time scales the electron inertia can be neglected ➔

Use the Boltzmann relation: : $n_e = n_0 e^{e\phi_1/T_e} \approx n_0 \left(1 + \frac{e\phi_1}{T_e}\right)$

Since the electrostatic potential ϕ_1 has now entered the game, use the Poisson equation:

$$\nabla \cdot \boldsymbol{E} = -\nabla^2 \phi = \frac{e}{4\pi\epsilon_0}(n_i - n_e)$$

Small perturbation: $n_i = n_0 + n_1$ ➔ $4\pi\epsilon_0 \nabla^2 \phi_1 = e\left(n_1 - n_0 \frac{e\phi}{T_e}\right)$

From derivatives to algebra …

FT everything (including the continuity equation) ➔

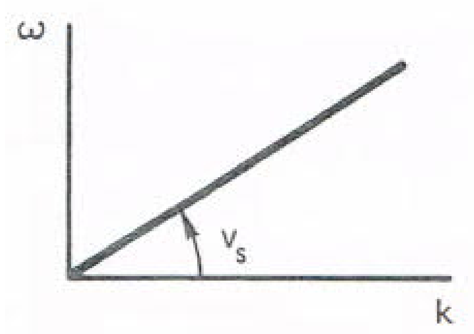
$$-i\omega n_0 M v_1 = -e n_0 ik\phi_1 - ik\gamma_i T_i n_1$$

$$4\pi\epsilon_0\phi_1\left[k^2 + \frac{n_0 e^2}{4\pi\epsilon_0 T_e}\right] = e n_1$$

$$n_1 = \frac{k n_0}{\omega} v_1$$

Recall $\frac{n_0 e^2}{4\pi\epsilon_0 T_e} = \frac{1}{\lambda_D^2}$ ➔ (HW) $\omega^2 = k^2\left[\frac{T_e}{M}\frac{1}{1+k^2\lambda_D^2} + \frac{\gamma_i T_i}{M}\right]$

Ion acoustic wave



Recall the definition of sound speed: $v_s \equiv \sqrt{T/m}$

So we have found an ion wave that propagates with velocity that is reminicent of a sound wave, *but with two temperatures*!

➔ the acoustic wave persists even if $T_i \rightarrow 0$! (Laboratory plasmas)

- Especially for ions it is very unlikely that we would have a wave with $\lambda < \lambda_D$ (recall the smallness of λ_D) ➔ $k^2\lambda_D^2 \ll 1$

➔ For most plasmas of interest: $\omega = kv_s$, $v_s^2 \equiv \left[\frac{T_e}{M} + \frac{\gamma_i T_i}{M}\right]$

Notes on acoustic wave & Co

Note 1:

- while plasma frequency depends only on the plasma density, the sound speed depends only on the plasma temperature!

Note 2:

- sound wave = 1D compression ➔ $\gamma_i = 3$.
- Meanwhile electrons assumed infinitely mobile

➔ equalize T everywhere

➔ isothermal ➔ $\gamma_e = 1$. (Was already assumed)

Note 3:

- unlike ordinary fluids, in plasmas ions can transmit vibrations even in the absence of collisions. This is due to electrostatic interaction.

Comparison of electron and ion waves

Electron plasma wave:

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2}k^2 v_{th}^2$$

- essentially a *constant-frequency* wave, w/ a correction from thermal motion

Ion acoustic wave (with $k^2\lambda_D^2 \ll 1$):

$$\omega = kv_s, \qquad v_s^2 \equiv \left[\frac{T_e}{M} + \frac{\gamma_i T_i}{M}\right]$$

- a *constant-velocity* wave, $\omega_{ph} = \omega_{gr}$, that needs $T \neq 0$. (Without thermal motion, electrons would perfectly shield the charge from ion bunching)

Ion acoustic wave w/ ”short” wavelength

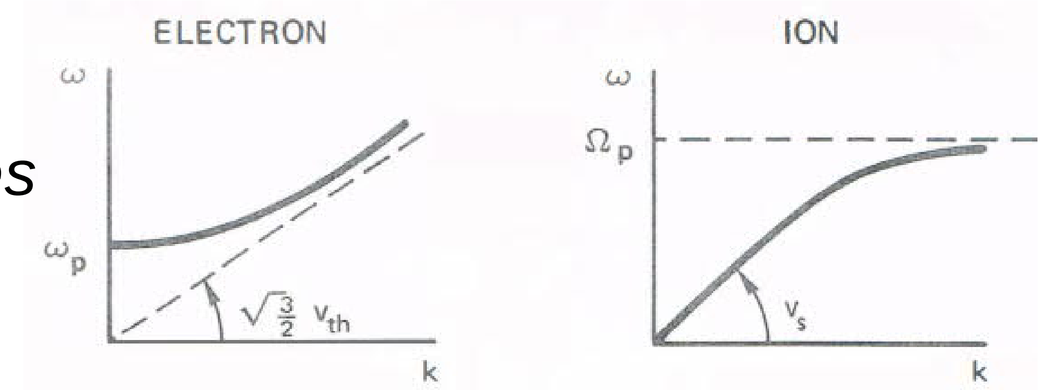
Now take the opposite limit: $k^2\lambda_D^2 \geq 1$, short(ish) wavelengths

$$\omega^2 \approx \frac{T_e}{M}\lambda_D^{-2} = \frac{n_0 e^2}{4\pi\epsilon_0 M} \equiv \omega_{pi}^2$$

So at short λ (high ω) the ion acoustic wave turns into constant-frequency oscillations!

Summa summarum:

electrostatic electron and ion waves have pretty much complementary behaviour.



And The Thing to remember …

You cannot extend the above treatment to arbitrarily high frequencies because our *initial assumption* was that the frequencies are *low* ($\ll \omega_{pe}$).

Never forget what your assumptions have been when you construct/extend a theoretical/numerical model !

Electrostatic waves in the presence of a background magnetic field

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Electric & magnetic fields
➔ review Maxwell's equations

$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$: Linearize & FT ➔ $\boldsymbol{k} \times \boldsymbol{E}_1 = i\omega \boldsymbol{B}_1$

Two cases:

$\boldsymbol{k} \parallel \boldsymbol{E}_1$ ➔ $\frac{\partial \boldsymbol{B}}{\partial t} = 0 \rightarrow \frac{\partial \boldsymbol{E}}{\partial t} = 0$ ➔ electro*static* wave

$\boldsymbol{k} \perp \boldsymbol{E}_1$ ➔ $\frac{\partial \boldsymbol{B}}{\partial t} \neq 0 \rightarrow \frac{\partial \boldsymbol{E}}{\partial t} \neq 0$ ➔ electro*magnetic* wave (next week)

Define two useful terms:

- *Longitudinal = parallel to* $\boldsymbol{k}$
- *Transverse = perpendicular to* $\boldsymbol{k}$

An external B-field ➔ preferred direction

Set axes so that $\boldsymbol{B} = \boldsymbol{B}_0\hat{\boldsymbol{z}}$

Two fundamentally different cases: $\boldsymbol{k} = k\hat{\boldsymbol{z}}$ or $\boldsymbol{k} \perp \boldsymbol{B}_0$

1. $\boldsymbol{k} = k\hat{\boldsymbol{z}}$

Since electrostatic motion is parallel to $\boldsymbol{k}$, the magnetic field does not have a say here

➔ the same results as for the $\boldsymbol{B} = 0$ case.

2. $\boldsymbol{k} \perp \boldsymbol{B}_0$

Here things get interesting because particles are not free to move across the magnetic field lines!

Electrostatic waves ⊥ *B*: $\boldsymbol{k} = k\hat{\boldsymbol{x}}$
-- high frequency

Immobile ions, only electrons have time to respond.

Take a stripped-down case: $n_0 = const.$, $\boldsymbol{v}_0 = 0$, cold plasma: $T = 0$

$$m\frac{\partial \boldsymbol{v}_1}{\partial t} = -e(\boldsymbol{E}_1 + \boldsymbol{v}_1 \times \boldsymbol{B}_0)$$

➔
$$-i\omega m v_x = -eE_1 - ev_y B_0$$
$$-i\omega m v_y = ev_x B_0$$
$$-i\omega m v_z = 0$$

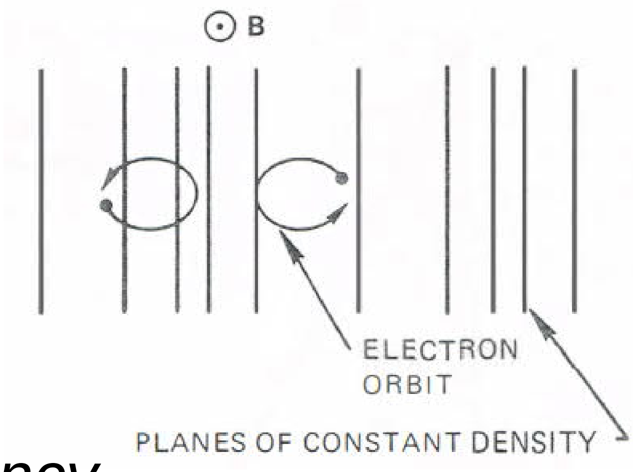
➔ $v_x = \frac{eE_1/im\omega}{1-\Omega_e^2/\omega^2}$; Note: v_x diverges when $\omega \to \Omega_e$!!??

Eliminating the rest of the junk

$\frac{\partial n_1}{\partial t} + n_0 \nabla \cdot (\boldsymbol{v}_1) = 0$ ➔ $n_1 = \frac{k}{\omega} n_0 v_x$

$\nabla \cdot \boldsymbol{E}_1 = -\frac{e n_1}{\epsilon_0}$ ➔ $ik\epsilon_0 E_1 = -e \frac{k}{\omega} n_0 \frac{eE_1/im\omega}{1-\Omega_e^2/\omega^2}$

➔ $\omega^2 = \omega_p^2 + \Omega_e^2 \equiv \omega_h^2$; the *upper hybrid frequency*



So the electrons oscillate faster than without B-field. This is because there are *two* restoring forces: the $\boldsymbol{E}_1$field generated by the perturbation, and the $\boldsymbol{v}_1 \times \boldsymbol{B}_0$ force.

Sanity checks:

- $\boldsymbol{B}_0 \rightarrow 0$: regular plasma oscillation at ω_p^2
- $n_0 \rightarrow 0$: simple gyromotion since the ES force vanishes

What about ion oscillations?

Here we cannot use the Boltzmann relation for electrons because the ion bunching is $\perp \boldsymbol{B}_0$ ➔ electrons are not free to follow …

➔ need to study both species. Use earlier eqs w/ $E_1 = -\nabla\phi_1$:

$$v_{i1} = \frac{ek}{M\omega}\phi_1\left(1-\frac{\Omega_i^2}{\omega^2}\right)^{-1} \quad \& \quad n_{i1} = n_0\frac{k}{\omega}v_{i1}\ ,\ \Omega_\mathrm{i} = qB_0/M$$

$$v_{e1} = -\frac{ek}{m\omega}\phi_1\left(1-\frac{\Omega_e^2}{\omega^2}\right)^{-1} \quad \& \quad n_{e1} = n_0\frac{k}{\omega}v_{e1}\ ,\ \Omega_\mathrm{e} = qB_0/m$$

Plasma approximation: $n_i = n_e \rightarrow v_{i1} = v_{e1}$

Lower hybrid oscillations

➔ $M\left(1-\frac{\Omega_i^2}{\omega^2}\right) = -m\left(1-\frac{\Omega_e^2}{\omega^2}\right)$

➔ … algebra … (HW)

➔ $\omega = \sqrt{\Omega_i \Omega_e} \equiv \omega_l$; *lower hybrid frequency*

LH waves can only be launched/observed VERY close to perpendicular. Even a small deviation changes the physics …

Ion waves not exactly perpendicular to B…

If $k_{\|} \neq 0$ ➔ electrons can swiftly (= within $1/\omega$) move to shield out the ion charge bunching

➔ neglect electron inertia = adopt Boltzmann relation: $n_{e1} = n_0 \frac{e\phi_1}{T_e}$

Plasma approximation ➔ $n_{i1} = n_0 \frac{e\phi_1}{T_e}$

Continuity equation ➔ $n_{i1} = n_0 \frac{k}{\omega} v_{i1}$

➔ $\left(1 - \frac{\Omega_i^2}{\omega^2}\right) v_{i1} = \frac{ek}{M\omega} \frac{T_e}{en_0} \frac{n_0 k}{\omega} v_{i1} = \frac{k^2}{\omega^2} v_s^2 v_{i1}$

Electrostatic ion cyclotron wave

The dispersion relation: $\omega^2 = \Omega_i^2 + k^2 v_s^2$

The physics of *electrostatic ion cyclotron wave*:

like the ion acoustic wave, but now the restoring force is enhanced by the $\boldsymbol{v}_1 \times \boldsymbol{B}_0$ force.