Notes - Theoretical exercises 4

December 3, 2021

Exercise 3.4 b)

How to compute $\text{Cov}(x_t x_{t-\tau})$?

$$\text{Cov}(x_t x_{t-\tau}) \overset{\mathbb{E}(x_t)=0}{=} \mathbb{E}(x_t x_{t-\tau}) = \mathbb{E}\left(\sum_{i=0}^{\infty} \phi^i \varepsilon_{t-i} \sum_{j=0}^{\infty} \phi^j \varepsilon_{t-\tau-j}\right)$$
$$\overset{\text{hint}}{=} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \phi^{i+j}\, \mathbb{E}(\varepsilon_{t-i}\varepsilon_{t-\tau-j}).$$

Use the fact that

$$\mathbb{E}(\varepsilon_{t-i}\varepsilon_{t-\tau-j}) = \begin{cases} \sigma^2, & t-i = t-\tau-j \\ 0, & t-i \neq t-\tau-j \end{cases}.$$

Then we get

$$\begin{aligned} &\text{Cov}(x_t x_{t-\tau}) = \phi^\tau \sigma^2 + \phi^{\tau+2}\sigma^2 + \phi^{\tau+4}\sigma^2 + \phi^{\tau+6}\sigma^2 + \ldots \\ &= \phi^\tau \sigma^2 (1 + \phi^2 + \phi^4 + \phi^6 + \ldots) \\ &= \phi^\tau \sigma^2 \sum_{i=0}^{\infty} \phi^{2i} = \phi^\tau \frac{\sigma^2}{1-\phi^2}. \end{aligned}$$

Exercise 4.1

a)

- Remember from lecture slides 3 that an AR(p) process is weakly stationary if and only if the roots of the lag polynomial (of the AR part) lie outside the unit circle.

- If it happens that a root is complex, e.g $L = a + bi$, $a, b \in \mathbb{R}$, then one has to check that
$$\sqrt{a^2 + b^2} > 1.$$
- If a root is real-valued, e.g. $L = c$, $c \in \mathbb{R}$, then it is sufficient to check that
$$|c| > 1.$$

b) Autocorrelation function for stationary AR(1) $x_t = \phi_1 x_{t-1} + \varepsilon_t$ process is
$$\rho(\tau) = \phi_1^\tau.$$
On the other hand, partial autocorrelation function for AR(1) process is
$$\alpha(\tau) = \begin{cases} 1, & \tau = 0 \\ \rho(1), & \tau = 1 \\ 0, & \tau \geq 2 \end{cases}.$$

Exercise 4.2

Choose $k = 2$ in Yule-Walker equations (See lecture slides 3). Then we get
$$\begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{21} \\ \alpha_{22} \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}.$$
Now solve α_{22}.

Exercise 4.3

i) MA(1)**:**
$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi}(\gamma_0 + 2\gamma_1 \cos(\lambda)) \\ &= \frac{1}{2\pi}((1 + \theta_1^2)\sigma^2 + 2\theta_1\sigma^2 \cos(\lambda)) \\ &= \frac{\sigma^2}{2\pi}(1 + \theta_1^2 + 2\theta_1 \cos(\lambda)). \end{aligned}$$

ii) $\text{SMA}(1)_{12}$**:**

$$\begin{aligned} f(\lambda) &= \frac{1}{2\pi}(\gamma_0 + 2\gamma_{12}\cos(12\lambda)) \\ &= \frac{1}{2\pi}((1+\Theta_1^2)\sigma^2 + 2\Theta_1\sigma^2\cos(12\lambda)) \\ &= \frac{\sigma^2}{2\pi}(1+\Theta_1^2 + 2\Theta_1\cos(12\lambda)). \end{aligned}$$

Hints for Exercise 4.4 (Homework)

- Tables about behavior of ACF/PACF for different SARMA models are useful. These tables are found in lecture slides 4.
- Spectral densities are given in the figures. However, one does not have to interpret them in order to solve the exercise.