5. Theoretical exercises

Demo exercises

Throughout these exercises, assume that $\mathbb{E}[x_{t-v}\epsilon_t] = 0$ for all $v \geq 1$. In addition, assume that $(\epsilon_t)_{t\in T} \sim$ i.i.d.$(0, \sigma^2)$, such that $\sigma^2 < +\infty$.

5.1 Consider the following ARMA processes:

$$x_t + \frac{3}{4}x_{t-1} = \epsilon_t - \epsilon_{t-1} \tag{1}$$

$$x_t + x_{t-2} = \epsilon_t - \frac{5}{6}\epsilon_{t-1} + \frac{1}{6}\epsilon_{t-2} \tag{2}$$

$$x_t - \epsilon_t - \frac{1}{16}x_{t-4} - \frac{4}{9}\epsilon_{t-2} = 0 \tag{3}$$

Which of the processes are (weakly) stationary? Which of the processes are invertible?

Solution. An ARMA process is stationary, if the zeros of the autoregressive polynomial lie outside the closed unit disk. An ARMA process is invertible, if the zeros of the moving average polynomial lie outside the closed unit disk.

(1) AR polynomial:

$$1 + \frac{3}{4}L = 0$$
$$L = -\frac{4}{3}$$
$$\left|-\frac{4}{3}\right| > 1$$

The process is stationary. MA polynomial:

$$1 - L = 0$$
$$L = 1$$

The process is not invertible.

(2) AR polynomial:

$$1 + L^2 = 0$$
$$L = \pm i$$
$$|L| = 1$$

The process is not stationary. MA polynomial:

$$1 - \frac{5}{6}L + \frac{1}{6}L^2 = 0$$
$$L = \frac{\frac{5}{6} \pm \sqrt{\frac{1}{36}}}{\frac{2}{6}} = 3 \quad \text{or} \quad 2$$

The process is invertible.

(3) AR polynomial:

$$\begin{aligned} 1-\frac{1}{16}L^4 &= 0 \\ L &= \pm 2 \quad \text{or} \\ L &= \pm 2i, \end{aligned}$$

which gives,

$$|L| = 2$$

and thus the process is stationary. MA polynomial:

$$\begin{aligned} 1+\frac{4}{9}L^2 &= 0 \\ L &= \pm\frac{3}{2}i \\ \left|\pm\frac{3}{2}i\right| &= \frac{3}{2} \end{aligned}$$

The process is invertible.

5.2 Let $\gamma(\cdot)$ be the autocovariance function of a weakly stationary process x_t. Show that the following properties hold.

(i) $\gamma(0) \geq 0$,

(ii) $|\gamma(\tau)| \leq \gamma(0)$, for every $\tau \in T$,

(iii) $\gamma(\tau) = \gamma(-\tau)$, for every $\tau \in T$.

Solution. Recall that a weakly stationary process x_t satisfies $\mathrm{Var}(x_t) < \infty$, for every $t \in T$. The autocovariance function of the process is

$$\gamma(\tau) = \mathrm{Cov}\left(x_t, x_{t-\tau}\right) = \mathbb{E}\left(\left(x_t - \mathbb{E}(x_t)\right)\left(x_{t-\tau} - \mathbb{E}(x_{t-\tau})\right)\right), \quad t, \tau \in T.$$

The first property holds, since

$$\gamma(0) = \mathrm{Var}(x_t) = \mathbb{E}\left(\left(x_t - \mathbb{E}(x_t)\right)^2\right) \geq 0.$$

The second property is obtained by using the Cauchy-Schwartz inequality.

$$\begin{aligned} &\left|\mathbb{E}\left(\left(x_t - E(x_t)\right)\left(x_{t-\tau} - \mathbb{E}(x_{t-\tau})\right)\right)\right|^2 \leq \mathbb{E}\left(\left(x_t - E(x_t)\right)^2\right)\mathbb{E}\left(\left(x_{t-\tau} - \mathbb{E}(x_{t-\tau})\right)^2\right) \\ &\Rightarrow \left|\mathrm{Cov}\left(x_t, x_{t-\tau}\right)\right|^2 \leq \mathrm{Var}(x_t)\mathrm{Var}(x_{t-\tau}) = \mathrm{Var}(x_t)^2 \\ &\Rightarrow |\gamma(\tau)| \leq \gamma(0), \end{aligned}$$

since the variance of a stationary process is time invariant, we have that,

$$\mathrm{Var}(x_t)\mathrm{Var}(x_{t-\tau}) = \mathrm{Var}(x_t)^2.$$

We get the third property from,

$$\gamma(-\tau) = \mathrm{Cov}\left(x_t, x_{t+\tau}\right) = \mathrm{Cov}\left(x_{t+\tau}, x_t\right) = \mathrm{Cov}\left(x_t, x_{t-\tau}\right) = \gamma(\tau),$$

since the autocovariance of a stationary process depends only on the time interval between the two random variables.

5.3 Derive the optimal 3-step prediction for the stationary AR(2) process

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \varepsilon_t, \quad \varepsilon_t \sim \mathrm{iid}(0, \sigma^2),$$

in the sense of mean squared error, when the process has been observed up to the time t. Assume that x_t and ε_s are independent when $s > t$. What is the recursive formula of s-step prediction?

Solution. The 1-step prediction is

$$\begin{aligned}\hat{x}_{t+1|t} &= \mathbb{E}\left(x_{t+1}|x_t, x_{t-1}, \ldots\right)\\ &= \mathbb{E}\left(\phi_1 x_t + \phi_2 x_{t-1} + \varepsilon_{t+1}|x_t, x_{t-1}, \ldots\right) = \phi_1 x_t + \phi_2 x_{t-1}.\end{aligned}$$

The 2-step prediction can be obtained by using the 1-step prediction.

$$\begin{aligned}\hat{x}_{t+2|t} &= \mathbb{E}\left(\phi_1 x_{t+1}|x_t, \ldots\right) + \mathbb{E}\left(\phi_2 x_t|x_t, \ldots\right) + 0\\ &= \phi_1\left(\phi_1 x_t + \phi_2 x_{t-1}\right) + \phi_2 x_t = \left(\phi_1^2 + \phi_2\right) x_t + \phi_1\phi_2 x_{t-1}.\end{aligned}$$

Similarly, the 3-step prediction is

$$\begin{aligned}\hat{x}_{t+3|t} &= \mathbb{E}\left(\phi_1 x_{t+2}|x_t, \ldots\right) + \mathbb{E}\left(\phi_2 x_{t+1}|x_t, \ldots\right)\\ &= \phi_1\left(\left(\phi_1^2 + \phi_2\right) x_t + \phi_1\phi_2 x_{t-1}\right) + \phi_1\phi_2 x_t + \phi_2^2 x_{t-1}\\ &= \left(\phi_1^3 + 2\phi_1\phi_2\right) x_t + \left(\phi_1^2\phi_2 + \phi_2^2\right) x_{t-1}.\end{aligned}$$

The s-step does not admit a closed form representation, but it can be written recursively as

$$\hat{x}_{t+s|t} = \phi_1 \hat{x}_{t-1+s|t} + \phi_2 \hat{x}_{t-2+s|t}.$$

Homework

5.4 Consider the following ARMA processes:

$$x_t - x_{t-1} + x_{t-2} = \epsilon_t + \epsilon_{t-1} - 6\epsilon_{t-2}, \tag{4}$$
$$x_t + \frac{1}{2}x_{t-1} = \epsilon_t + \frac{4}{3}\epsilon_{t-1} + \frac{1}{3}\epsilon_{t-2}, \tag{5}$$
$$x_t - x_{t-1} = \epsilon_t + \frac{1}{2}\epsilon_{t-12}. \tag{6}$$

Which of the processes are stationary? Which of the processes are invertible?

5.5 Derive the optimal s-step prediction for the invertible MA(q) process,

$$x_t = \sum_{i=0}^{q} \theta_i L^i \varepsilon_t, \quad \varepsilon_t \sim \text{iid}(0, \sigma^2),$$
$$\theta_0 = 1,$$

in the sense of mean squared error, when the process ε_t has been observed up to point of time t.