Theoretical Exercises 5
Prediction and Time Series Analysis

November 30, 2021

5.1

Weak stationarity and invertibility can be checked by examining the roots of the lag polynomials of the AR and MA parts of the model. (Weak) stationarity is equivalent to the roots of the AR lag polynomial being outside the unit disk in the complex plane whereas invertibility is equivalent to the MA lag polynomial roots having the aforementioned property. Some technical assumptions are needed to ensure the equivalences but they are not important for this course.

$$
\begin{aligned}
& x_t + \frac{3}{4}x_{t-1} = \epsilon_t - \epsilon_{t-1} \iff (1+\frac{3}{4}L)x_t = (1-L)\epsilon_t \qquad (1)\\
& \text{AR part: } 1+\frac{3}{4}L = 0 \iff L = -\frac{4}{3}, |L| = \frac{4}{3} > 1 \Rightarrow \text{Process (1) is stationary}\\
& \text{MA part: } 1-L=0 \iff L=1, |L|=1 \Rightarrow \text{Process (1) is not invertible}
\end{aligned}
$$

$$
\begin{aligned}
& x_t + x_{t-2} = \epsilon_t - \frac{5}{6}\epsilon_{t-1} + \frac{1}{6}\epsilon_{t-2} \iff (1+L^2)x_t = (1-\frac{5}{6}L+\frac{1}{6}L^2)\epsilon_t \qquad (2)\\
& \text{AR part: } 1+L^2=0 \iff L^2=-1 \iff L=\pm\sqrt{-1}=\pm i = 0+(\pm 1)i, |L| = \sqrt{0+(\pm 1)^2} = 1\\
& \Rightarrow \text{Process (2) is not stationary}\\
& \text{MA part: } 1-\frac{5}{6}L+\frac{1}{6}L^2 = 0 \iff L^2-5L+6=0 \iff L = \frac{5\pm\sqrt{25-4\times 6}}{2} = \frac{5\pm 1}{2} = \begin{cases} 3, + \\ 2, - \end{cases}\\
& |L| > 1 \Rightarrow \text{Process (2) is invertible}
\end{aligned}
$$

$$
\begin{aligned}
& x_t - \frac{1}{16}x_{t-4} = \epsilon_t + \frac{4}{9}\epsilon_{t-2} \iff (1-\frac{1}{16}L^4)x_t = (1+\frac{4}{9}L^2)\epsilon_t \qquad (3)\\
& \text{AR part: } 1-\frac{1}{16}L^4 = 0 \iff L^4 = 16 \text{ We really only care about the modulus of the roots so}\\
& |L^4| = |L|^4 = 16 \iff |L| = 2 > 1 \Rightarrow \text{Process (3) is stationary (The roots are } \pm 2, \pm 2i)\\
& \text{MA part: } 1+\frac{4}{9}L^2 = 0 \iff L^2 = -\frac{9}{4} \iff L = \pm\frac{3}{2}i, |L| = \frac{3}{2} > 1 \Rightarrow \text{Process (3) is invertible}
\end{aligned}
$$

5.2

Definition:

$$\gamma : \mathbb{Z} \to \mathbb{R}, \gamma(k) = \mathrm{Cov}[x_t, x_{t-k}] = \mathbb{E}[(x_t-\mu)(x_{t-k}-\mu)] \qquad (4)$$

where $\mu = \mathbb{E}[x_t]\ \forall\ t$ is the shared expected value of the weakly stationary process.

Property (i):

$$\gamma(0) = \mathrm{Var}[x_t] = \mathbb{E}[(x_t-\mu)^2] \geq \mathbb{E}[0] = 0$$

Property (ii): Covariance can be interpreted as an inner product in a certain vector space of random variables (those with a finite second moment). Thus it satisfies the Cauchy-Schwarz inequality

$$|\langle x, y\rangle| \leq ||x||||y|| \iff \langle x, y\rangle^2 \leq \langle x, x\rangle\langle y, y\rangle \ \forall\ x, y \tag{5}$$

where $\langle \cdot, \cdot \rangle$ is an inner product and $|| \cdot ||$ the norm induced by the inner product. Thus

$$\gamma(k)^2 = \text{Cov}[x_t, x_{t-k}]^2 \leq \text{Cov}[x_t, x_t]\text{Cov}[x_{t-k}, x_{t-k}] = \text{Var}[x_t]\text{Var}[x_{t-k}] = \text{Var}[x_t]^2$$
$$\Rightarrow |\gamma(k)| \leq \gamma(0)$$

where we used the fact that the variance of x_t is constant across time.

Property (iii):

$$\gamma(k) = \text{Cov}[x_t, x_{t-k}] = \text{Cov}[x_{t-k}, x_t] = \text{Cov}[x_t, x_{t+k}] = \gamma(-k)$$

where we used the symmetry of covariance and the fact that the covariance of a stationary process depends only on the lag between the time points.

5.3

The (conditional) expected value can be shown to be optimal in the sense of minimizing the mean squared prediction error $\mathbb{E}[(x_{t+s} - \hat{x}_{t+s})^2 | x_t, x_{t-1}, \ldots]$. Thus $\hat{x}_{t+s} = \hat{x}_{t+s|t} = \mathbb{E}[x_{t+s} | x_t, x_{t-1}, \ldots]$.

One-step prediction:

$$\hat{x}_{t+1|t} = \mathbb{E}[x_{t+1}|x_t, \ldots] = \mathbb{E}[\phi_1 x_t + \phi_2 x_{t-1} + \epsilon_{t+1}| \ldots]$$
$$= \phi_1 \mathbb{E}[x_t| \ldots] + \phi_2 \mathbb{E}[x_{t-1}| \ldots] + \mathbb{E}[\epsilon_{t+1}| \ldots]$$

Note that the conditional expected value is linear just like the expected value. A couple of useful properties are needed to proceed. The conditional expected value of a random variable when conditioned on itself is just the original random variable. The conditional expected value of a random variable is just the expected value when it and the random variables in the condition expression are independent. Thus

$$\hat{x}_{t+1|t} = \phi_1 \mathbb{E}[x_t| \ldots] + \phi_2 \mathbb{E}[x_{t-1}| \ldots] + \mathbb{E}[\epsilon_{t+1}| \ldots] = \phi_1 x_t + \phi_2 x_{t-1} + \mathbb{E}[\epsilon_{t+1}]$$
$$= \phi_1 x_t + \phi_2 x_{t-1} \tag{6}$$

Two-step prediction:

$$\hat{x}_{t+2|t} = \mathbb{E}[x_{t+2}|x_t, \ldots] = \phi_1 \mathbb{E}[x_{t+1}| \ldots] + \phi_2 \mathbb{E}[x_t| \ldots] = \phi_1(\phi_1 x_t + \phi_2 x_{t-1}) + \phi_2 x_t = (\phi_1^2 + \phi_2)x_t + \phi_1\phi_2 x_{t-1} \tag{7}$$

Three-step prediction:

$$\hat{x}_{t+3|t} = \mathbb{E}[x_{t+3}|x_t, \ldots] = \phi_1 \mathbb{E}[x_{t+2}| \ldots] + \phi_2 \mathbb{E}[x_{t+1}| \ldots]$$
$$= \phi_1((\phi_1^2 + \phi_2)x_t + \phi_1\phi_2 x_{t-1}) + \phi_2(\phi_1 x_t + \phi_2 x_{t-1}) = (\phi_1^3 + 2\phi_1\phi_2)x_t + (\phi_1^2\phi_2 + \phi_2^2)x_{t-1} \tag{8}$$

From the expressions, we can observe that the general s-step prediction satisfies the following recursive identity.

$$\hat{x}_{t+s|t} = \phi_1 \hat{x}_{t-1+s|t} + \phi_2 \hat{x}_{t-2+s|t} \tag{9}$$

Hints for 5.4

Very similar to 5.1. Remember that we only care about whether the roots of the lag polynomials are outside the unit disk or not.

Hints for 5.5

To figure out a general expression for the optimal s-step prediction, you can start by considering 1-step and 2-step predictions and extrapolating from those. The properties of the conditional expected value mentioned in 5.3 will be useful.