Notes - Theoretical exercises 5

November 28, 2021

Exercise 5.2

- Cauchy-Schwarz inequality for square integrable random variables X and Y is
$$|\,\mathbb{E}(XY)|^2 \leq \mathbb{E}(X^2)\,\mathbb{E}(Y^2).$$

A random variable X is square integrable if $\mathbb{E}(X^2) < \infty$.

Exercise 5.3

- 1-step prediction:

$$\begin{aligned}\hat{x}_{t+1} &= \mathbb{E}(x_{t+1}|x_t, x_{t-1}, \ldots) = \mathbb{E}(\phi_1 x_t + \phi_2 x_{t-1} + \varepsilon_{t+1}|x_t, x_{t-1}, \ldots)\\ &= \phi_1\,\mathbb{E}(x_t|x_t, x_{t-1}, \ldots) + \phi_2\,\mathbb{E}(x_{t-1}|x_t, x_{t-1}, \ldots) + \mathbb{E}(\varepsilon_{t+1}|x_t, x_{t-1}, \ldots)\\ &= \phi_1 x_t + \phi_2 x_{t-1} + \mathbb{E}(\varepsilon_{t+1}) = \phi_1 x_t + \phi_2 x_{t-1}.\end{aligned}$$

- 2-step prediction:

$$\begin{aligned}\hat{x}_{t+2} &= \mathbb{E}(x_{t+2}|x_t, x_{t-1}, \ldots) = \mathbb{E}(\phi_1 x_{t+1} + \phi_2 x_t + \varepsilon_{t+2}|x_t, x_{t-1}, \ldots)\\ &= \phi_1\,\mathbb{E}(x_{t+1}|x_t, x_{t-1}, \ldots) + \phi_2\,\mathbb{E}(x_t|x_t, x_{t-1}, \ldots) + \mathbb{E}(\varepsilon_{t+2}|x_t, x_{t-1}, \ldots)\\ &= \phi_1\hat{x}_{t+1} + \phi_2 x_t + \mathbb{E}(\varepsilon_{t+2}) = \phi_1(\phi_1 x_t + \phi_2 x_{t-1}) + \phi_2 x_t.\end{aligned}$$

- 3-step prediction:

$$\begin{aligned}\hat{x}_{t+3} &= \mathbb{E}(x_{t+3}|x_t, x_{t-1}, \ldots) = \mathbb{E}(\phi_1 x_{t+2} + \phi_2 x_{t+1} + \varepsilon_{t+3}|x_t, x_{t-1}, \ldots)\\ &= \phi_1\,\mathbb{E}(x_{t+2}|x_t, x_{t-1}, \ldots) + \phi_2\,\mathbb{E}(x_{t+1}|x_t, x_{t-1}, \ldots) + \mathbb{E}(\varepsilon_{t+3}|x_t, x_{t-1}, \ldots)\\ &= \phi_1\hat{x}_{t+2} + \phi_2\hat{x}_{t+1} + \mathbb{E}(\varepsilon_{t+3}) = \phi_1\hat{x}_{t+2} + \phi_2\hat{x}_{t+1}.\end{aligned}$$

- s-step prediction:

$$\begin{aligned}\hat{x}_{t+s} &= \mathbb{E}(x_{t+s}|x_t, x_{t-1}, \ldots) = \mathbb{E}(\phi_1 x_{t+s-1} + \phi_2 x_{t+s-2} + \varepsilon_{t+s}|x_t, x_{t-1}, \ldots) \\ &= \phi_1 \mathbb{E}(x_{t+s-1}|x_t, x_{t-1}, \ldots) + \phi_2 \mathbb{E}(x_{t+s-2}|x_t, x_{t-1}, \ldots) + \mathbb{E}(\varepsilon_{t+s}|x_t, x_{t-1}, \ldots) \\ &= \phi_1 \hat{x}_{t+s-1} + \phi_2 \hat{x}_{t+s-2} + \mathbb{E}(\varepsilon_{t+s}) = \phi_1 \hat{x}_{t+s-1} + \phi_2 \hat{x}_{t+s-2}.\end{aligned}$$

Exercise 5.4 (Homework)

This exercise is very similar to Exercise 5.1.

Exercise 5.5 (Homework)

It is maybe useful to first think about 1-step prediction

$$\hat{x}_{t+1} = \mathbb{E}(x_{t+1}|\varepsilon_t, \varepsilon_{t-1}, \ldots),$$

and 2-step prediction

$$\hat{x}_{t+2} = \mathbb{E}(x_{t+2}|\varepsilon_t, \varepsilon_{t-1}, \ldots),$$

and then try to figure out s-step prediction.