6. Theoretical exercises

Demo exercises

6.1 Show that the optimal mean squared error prediction for the stationary and invertible ARIMA(0,1,1) process,

$$Dx_t = \varepsilon_t + \theta_1 \varepsilon_{t-1}, \quad (\varepsilon_t)_{t\in T} \sim \mathrm{iid}\big(0, \sigma^2\big), \tag{1}$$

where ε_s and x_t are independent for $s > t$, satisfies the formula,

$$\hat{x}_{t+1|t} = \alpha x_t + (1-\alpha)\hat{x}_{t|t-1},$$

of exponential smoothing when $|\theta_1| < 1$ and $\alpha = 1 + \theta_1$.

Solution. We consider the optimal mean squared prediction of the ARIMA(0,1,1) process,

$$\hat{x}_{t+1|t} = \mathbb{E}\left[x_{t+1} \mid \varepsilon_t, \varepsilon_{t-1}, ...\right].$$

Since

$$Dx_t = x_t - x_{t-1} = \varepsilon_t + \theta_1 \varepsilon_{t-1}, \quad t \in T,$$

we have that,

$$x_{t+1} = x_t + \varepsilon_{t+1} + \theta_1 \varepsilon_t.$$

The optimal prediction in the sense of the mean squared error is

$$\begin{aligned}\hat{x}_{t+1|t} &= \mathbb{E}\big[x_{t+1} \mid \varepsilon_t, \varepsilon_{t-1}, ...\big] = \mathbb{E}\big[x_t + \varepsilon_{t+1} + \theta_1\varepsilon_t \mid \varepsilon_t, \varepsilon_{t-1}, ...\big] \\ &= x_t + \theta_1 \varepsilon_t.\end{aligned}$$

Then, by combining Equation (1) with $\hat{x}_{t|t-1} = x_{t-1} + \theta_1\varepsilon_{t-1}$, we get that $\varepsilon_t = x_t - \hat{x}_{t|t-1}$. Thus,

$$\begin{aligned}\hat{x}_{t+1|t} &= x_t + \theta_1\varepsilon_t = x_t + \theta_1\big(x_t - \hat{x}_{t|t-1}\big) \\ &= (1+\theta_1)x_t - \theta_1\hat{x}_{t|t-1} = \alpha x_t + (1-\alpha)\hat{x}_{t|t-1},\end{aligned}$$

which concludes the proof.

6.2 Show that ARMA(p, q) process

$$\begin{aligned}\Phi(L)y_t &= \Theta(L)\varepsilon_t, \quad (\varepsilon_t)_{t\in T} \sim \mathrm{WN}\left(0, \sigma^2\right),\\ \Phi(L) &= 1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_p L^p,\\ \Theta(L) &= 1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q,\end{aligned}$$

has the following state–space representation,

$$\begin{aligned}x_{t+1} &= \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{r-1} & \phi_r \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} x_t + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\\ y_t &= \begin{bmatrix} 1 & \theta_1 & \theta_2 \cdots \theta_{r-1} \end{bmatrix} x_t,\end{aligned}$$

where $r = \max\{p, q+1\}$ and

$$\phi_j = 0, \quad \text{when } j > p \quad \text{and} \quad \theta_j = 0 \quad \text{when } j > q.$$

Solution.

The ARMA(p, q) process can be expressed as,

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \ldots + \phi_r y_{t-r} + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_{r-1}\varepsilon_{t-r+1}, \tag{2}$$

where $r = \max\{p, q+1\}$ and $\phi_j = 0$, when $j > p$, and $\theta_j = 0$, when $j > q$. The corresponding lag polynomial representation is

$$(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_r L^r)y_t = (1 + \theta_1 L + \theta_2 L^2 + \ldots \theta_{r-1} L^{r-1})\varepsilon_t.$$

The objective is to show that the state space representation corresponds to Equation (2). The first row of the state space representation gives

$$x_{t+1,1} = \phi_1 x_{t,1} + \phi_2 x_{t,2} + \ldots + \phi_r x_{t,r} + \varepsilon_{t+1}, \tag{3}$$

where $x_t = (x_{t,1}, \ldots, x_{t,r})^\top$. The second row gives

$$x_{t+1,2} = x_{t,1}.$$

By the equation above, the third row can be written as

$$x_{t+1,3} = x_{t,2} = x_{t-1,1}.$$

Using the same logic, we get for the j:th row, $j \geq 2$, that,

$$x_{t+1,j} = L^{j-1} x_{t+1,1} = x_{t+2-j,1},$$

which can be used to reformulate the first row of the state space presentation as

$$x_{t+1,1} = \left(\phi_1 + \phi_2 L + \phi_3 L^2 + \ldots + \phi_r L^{r-1}\right) x_{t,1} + \varepsilon_{t+1},$$

which is equivalent with,

$$\varepsilon_{t+1} = \left(1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3 + \ldots - \phi_r L^r\right) x_{t+1,1}. \tag{4}$$

The observation equation is of the form

$$y_t = x_{t,1} + \theta_1 x_{t,2} + \ldots + \theta_{r-1} x_{t,r} = (1 + \theta_1 L + \ldots + \theta_{r-1} L^{r-1}) x_{t,1} \tag{5}$$

By multiplying the observation equation (5) with $(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_r L^r)$ and utilizing Equation (4), we obtain

$$\begin{aligned}(1 - \phi_1 L - \ldots - \phi_r L^r)\, y_t &= (1 + \theta_1 L + \ldots + \theta_{r-1} L^{r-1})\left(1 - \phi_1 L - \phi_2 L^2 - \ldots - \phi_r L^r\right) x_{t,1} \\ &= (1 + \theta_1 L + \ldots + \theta_{r-1} L^{r-1}) \varepsilon_t,\end{aligned}$$

which corresponds to the lag polynomial representation of the ARMA(p, q) process.