Notes - Theoretical exercises 6

December 2, 2021

Exercise 6.1

From the definition of process ARIMA$(0, 1, 1)$ we get

$$x_t = x_{t-1} + \varepsilon_t + \theta_1 \varepsilon_{t-1}. \tag{1}$$

Then

$$\begin{aligned} &\hat{x}_{t+1|t} = \mathbb{E}(x_{t+1}|\varepsilon_t, \varepsilon_{t-1}, \ldots) \\ &= \mathbb{E}(x_t + \varepsilon_{t+1} + \theta_1 \varepsilon_t|\varepsilon_t, \varepsilon_{t-1}, \ldots) \\ &= \mathbb{E}(x_t|\varepsilon_t, \varepsilon_{t-1}, \ldots) + \mathbb{E}(\varepsilon_{t+1}|\varepsilon_t, \varepsilon_{t-1}, \ldots) + \theta_1 \mathbb{E}(\varepsilon_t|\varepsilon_t, \varepsilon_{t-1}, \ldots) \\ &= x_t + \mathbb{E}(\varepsilon_{t+1}) + \theta_1 \varepsilon_t \\ &= x_t + \theta_1 \varepsilon_t. \end{aligned} \tag{2}$$

By combining Equation (1) and (2) we have

$$\varepsilon_t = x_t - \underbrace{(x_{t-1} + \theta_1 \varepsilon_{t-1})}_{=\hat{x}_{t|t-1}} = x_t - \hat{x}_{t|t-1}.$$

Then by combining above and Equation (2) we have

$$\begin{aligned} &\hat{x}_{t+1|t} = x_t + \theta_1 \varepsilon_t = x_t + \theta_1 (x_t - \hat{x}_{t|t-1}) \\ &= x_t + \theta_1 x_t - \theta_1 \hat{x}_{t|t-1} = (1 + \theta_1) x_t - \theta_1 \hat{x}_{t|t-1}. \end{aligned}$$

By choosing $\alpha = 1 + \theta_1$ we get the result.

Exercise 6.2

Vector x_t is defined as

$$x_t = (x_{t,1}, \ldots, x_{t,r})^T.$$

Then state-space representation for x_{t+1} is

$$\begin{bmatrix} x_{t+1,1} \\ x_{t+1,2} \\ x_{t+1,3} \\ \vdots \\ x_{t+1,r} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_{r-1} & \phi_r \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \begin{bmatrix} x_{t,1} \\ x_{t,2} \\ x_{t,3} \\ \vdots \\ x_{t,r} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_{t+1,1} \\ x_{t+1,2} \\ x_{t+1,3} \\ \vdots \\ x_{t+1,r} \end{bmatrix} = \begin{bmatrix} \phi_1 x_{t,1} + \phi_2 x_{t,2} + \dots + \phi_{r-1} x_{t,r-1} + \phi_r x_{t,r} + \varepsilon_{t+1} \\ x_{t,1} \\ x_{t,2} \\ \vdots \\ x_{t,r-1} \end{bmatrix}.$$

On the other hand, the state space representation for y_t is

$$y_t = \begin{bmatrix} 1 & \theta_1 & \theta_2 \dots \theta_{r-1} \end{bmatrix} \begin{bmatrix} x_{t,1} \\ x_{t,2} \\ \vdots \\ x_{t,r} \end{bmatrix}$$

$$\Rightarrow y_t = x_{t,1} + \theta_1 x_{t,2} \dots + \theta_{r-1} x_{t,r}.$$

Let's look at the state-space represenation of x_{t+1}. From there we get

$$\begin{aligned} &x_{t+1,2} = x_{t,1}, \\ &x_{t+1,3} = x_{t,2} = x_{t-1,1}, \\ &\vdots \\ &x_{t+1,r} = x_{t,r-1} = \dots = x_{t-r+1,1}. \end{aligned}$$

Thus

$$x_{t+1,i} = L^{j-1} x_{t+1,1} \quad \text{for } j = 2, \dots, r. \tag{3}$$

Then by Equation (3) we have

$$\begin{aligned} x_{t+1,1} &= (\phi_1 + \phi_2 L + \phi_3 L^2 + \dots + \phi_{r-1} L^{r-2} + \phi_r L^{r-1}) x_{t,1} + \varepsilon_{t+1} \\ \Rightarrow \varepsilon_{t+1} &= x_{t+1,1} - (\phi_1 + \phi_2 L + \phi_3 L^2 + \dots + \phi_{r-1} L^{r-2} + \phi_r L^{r-1}) x_{t,1} \\ \Rightarrow \varepsilon_{t+1} &= (1 - \phi_1 L - \phi_2 L^2 - \phi_3 L^3 - \dots - \phi_{r-1} L^{r-1} - \phi_r L^r) x_{t+1,1} \end{aligned}$$

and

$$y_t = (1 + \theta_1 L + \theta_2 L^2 + \dots + \theta_{r-2} L^{r-3} + \theta_{r-1} L^{r-1}) x_{t,1}.$$

By multiplying above with the lag polynomial

$$(1 - \phi_1 L - \phi_2 L^2 - \dots - \phi_{r-1} L^{r-1} - \phi_r L^r)$$

we get the result. Details can be found in the model solution.