PHYS-E055101 Low Temperature Physics: Nanoelectronics

Superconducting devices

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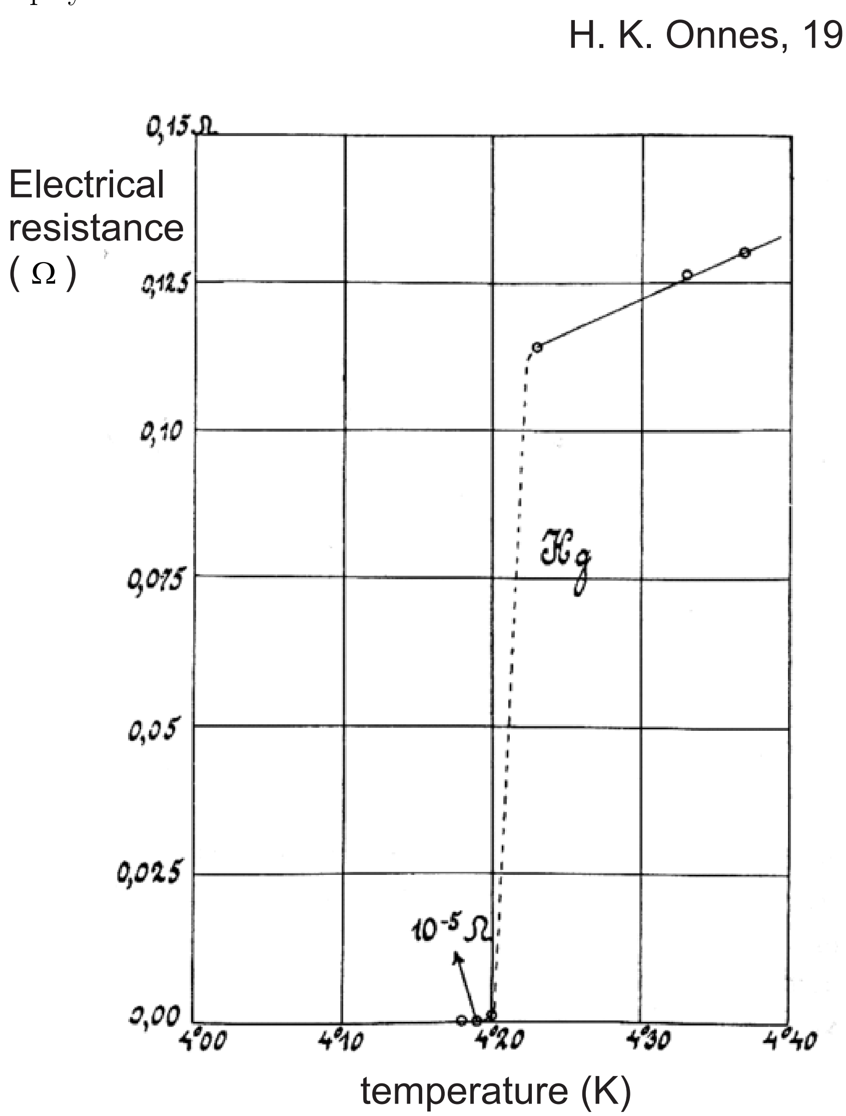
I. SUPERCONDUCTIVITY

What is superconductivity? It is a phenomenon occurring in certain materials at low temperatures, characterized by exactly-zero electrical resistance as one of the measurable signatures. Superconductivity was discovered in 1911 in the famous laboratory of Kamerlingh Onnes, where it was first noticed that the resistivity of Hg (mercury, a metal) drops abruptly to zero at around 4 K.



Is this surprising? Yes: the electrical resistivity of metals decreases with temperature, but it is never zero, due to impurities (electrons would scatter on those). However, for a superconductor, the resistance drops abruptly to exactly zero at a certain finite temperature. This suggests a phase transition.

So, isn't this some kind of perfect conductor - can it be for example understood as having zero impurities? It is actually much more than this. It cannot be understood in the framework of classical physics - *i.e.* classical electricity and magnetism. It has many other unusual properties: it expels the lines of magnetic field – even if these fields are static (Meissner effect). An electrical current flowing in in a superconducting ring will have no dissipation (these are called "persistent currents"), and as a result it will rotate there forever *i.e.* longer than the age of the Universe!

II. BCS THEORY OF SUPERCONDUCTIVITY

The microscopic theory that describes superconductivity is due to Bardeen, Cooper, and Schrieffer (BCS, 1957). The theory refers to s-wave superconductors (the pairing of the electrons is of s-type). Other types of superconductors (called high-Tc and believed to have d-wave pairing) have been discovered since - the first by Bednorz and Mueller (1986) - with critical temperatures that can exceed 100K.

The basic mechanism of superconductivity is the pairing of electrons, which will tend to form Cooper pairs. Pairing is due to an effective attraction between electrons - therefore the theory of superconductivity has to go beyond the noninteracting electron gas model and include interactions between electrons. It might come as a surprise that the e-e interaction is attractive: shouldn't be exactly the opposite be the case, since electrons have all negative charge there will be a repulsive force between them? Note that in the discussion of metals we have ignored completely the dynamics of the ionic background in which the electrons move: we have assumed that there exists some positive background which neutralizes the electron charge. However, consider what happens when one electron moves through the lattice. The nearby ions will create a positive charge density to compensate the negative charge of the electron. However, the speed of the electron is $v_{\rm F} = \hbar k_{\rm F}/m$; in contrast, the ions are heavier and therefore much slower. By the time the ions have polarized, the electron is long gone from that region. Now, if a second electron enters that region, it will experience a positive charge excess, to which it will be attracted. Note the long-range character of this $e-e$ interaction: the electrons can move far apart from each other, still feeling the effect of this interaction. Thus they do not form real molecules. This is reflected, as we will see, very accurately in the BCS theory.

The Hamiltonian for a system of interacting electrons is

$$\hat{H} = \sum_{k,\sigma} \left(\frac{\hbar^2 k^2}{2m} - \mu \right) \hat{c}^\dagger_{k,\sigma} \hat{c}_{k,\sigma} + \sum_{k,k',q,\sigma,\sigma'} V_{k,k',q,\sigma,\sigma'} \hat{c}^\dagger_{k,\sigma} \hat{c}^\dagger_{-k+q,\sigma'} \hat{c}_{-k'+q,\sigma'} \hat{c}_{k',\sigma}. \tag{1}$$

We then make two assumptions:

1) the electrons that pair up will have opposite spin (s-wave pairing), $\sigma = -\sigma'$.

2) the center of mass momentum of a pair is zero, that is, pairing occurs only for $q = 0$.

Formally, we can write $V_{k,k',q,\sigma,\sigma'} = (1/2) V_{k,k'} \delta_{q,0} \delta_{\sigma,-\sigma'}$, therefore

$$\hat{H} = \sum_{k,\sigma} \left(\frac{\hbar^2 k^2}{2m} - \mu \right) \hat{c}^\dagger_{k\sigma} \hat{c}_{k\sigma} + \sum_{k,k'} V_{k,k'} \hat{c}^\dagger_{k\uparrow} \hat{c}^\dagger_{-k\downarrow} \hat{c}_{-k'\downarrow} \hat{c}_{k'\uparrow}, \tag{2}$$

Next, we aim at applying the idea of pairing and treat the corresponding operators using mean-field ideas,

$$\hat{c}_{-k\downarrow} \hat{c}_{k\uparrow} = F_k + (\hat{c}_{-k\downarrow} \hat{c}_{k\uparrow} - F_k), \tag{3}$$

$$\hat{c}^\dagger_{k\uparrow} \hat{c}^\dagger_{-k\downarrow} = F^*_k + (\hat{c}^\dagger_{k\uparrow} \hat{c}^\dagger_{-k\downarrow} - F^*_k). \tag{4}$$

Here we defined the average of the pairing operator as

$$F_k = \langle \hat{c}_{-k,\downarrow} \hat{c}_{k,\uparrow} \rangle. \tag{5}$$

We will then treat the fluctuations around this average as small, in other words we write

$$\hat{c}^\dagger_{k\uparrow} \hat{c}^\dagger_{-k\downarrow} \hat{c}_{-k'\downarrow} \hat{c}_{k'\uparrow} = [F^*_k + (\hat{c}^\dagger_{k\uparrow} \hat{c}^\dagger_{-k\downarrow} - F^*_k)][F_{k'} + (\hat{c}_{-k'\downarrow} \hat{c}_{k'\uparrow} - F_{k'})] \tag{6}$$

$$\approx \hat{c}^\dagger_{k\uparrow} \hat{c}^\dagger_{-k\downarrow} F_{k'} + \hat{c}_{-k'\downarrow} \hat{c}_{k'\uparrow} F^*_k - F_{k'} F^*_k. \tag{7}$$

This results in the so-called BCS Hamiltonian,

$$H = \sum_{k,\sigma} \left(\frac{\hbar^2 k^2}{2m} - \mu \right) \hat{c}^\dagger_{k\sigma} \hat{c}_{k\sigma} + \sum_{k,k'} V_{k,k'} \left(\hat{c}^\dagger_{k\uparrow} \hat{c}^\dagger_{-k\downarrow} F_{k'} + \hat{c}_{-k'\downarrow} \hat{c}_{k'\uparrow} F^*_k - F_{k'} F^*_k \right). \tag{8}$$

Next, let us define

$$\xi_k = \frac{\hbar^2 k^2}{2m} - \mu, \tag{9}$$

and the so-called **superconducting gap**,

$$\Delta_k = -\sum_{k'} V_{kk'} \langle \hat{c}_{-k'\downarrow} \hat{c}_{k'\uparrow} \rangle = -\sum_{k'} V_{kk'} F_{k'}. \tag{10}$$

where we have put a minus sign to put in evidence the attractive character of the interaction. This results in the so-called **BCS-model Hamiltonian**,

$$H = \sum_{k,\sigma} \xi_k \hat{c}^\dagger_{k\sigma}\hat{c}_{k\sigma} - \sum_k \left(\Delta_k \hat{c}^\dagger_{k\uparrow}\hat{c}^\dagger_{-k\downarrow} + \Delta^*_k \hat{c}_{-k\downarrow}\hat{c}_{k\uparrow} - \Delta_k F^*_k\right). \tag{11}$$

This is already a nice result, from which you can see plenty of things: for example that the electrons are created and annihilated in pairs, with Δ_k playing the role of a pairing energy. Also, because Δ_k is defined as an average over the pairing operator, it has the significance of an order parameter.

Next, we would like to diagonalize the BCS Hamiltonian. To do so, we note that the Hamiltonian is quadratic, and that instead of the desired diagonal terms $\hat{c}^\dagger\hat{c}$ it also has mixed in the pairing operators $\hat{c}\hat{c}$ and $\hat{c}^\dagger\hat{c}^\dagger$. We would like to "unmix" these terms, and for this we should find a linear transformation that does precisely that. This transformation is called the **Bogoliubov (or Bogoliubov-Valatin) transformation**. It has the form

$$\hat{c}_{k\uparrow} = u^*_k \hat{\gamma}_{k0} + v_k \hat{\gamma}^\dagger_{k1}, \tag{12}$$

$$\hat{c}^\dagger_{-k\downarrow} = -v^*_k \hat{\gamma}_{k0} + u_k \hat{\gamma}^\dagger_{k1}, \tag{13}$$

where γ_{k0} and γ_{k1} are called quasiparticle operators. Remember that $\hat{c}$ are fermionic operators, that is

$$\{\hat{c}_{k,\sigma}, \hat{c}^\dagger_{k',\sigma'}\} = \delta_{kk'}\delta_{\sigma\sigma'}, \tag{14}$$

and so are the quasiparticle operators $\hat{\gamma}$,

$$\{\hat{\gamma}_{k,j}, \hat{\gamma}^\dagger_{k',j'}\} = \delta_{kk'}\delta_{jj'}, j, j' \in \{0,1\}. \tag{15}$$

Exercise: Show that this transformation is canonical (preserves the commutation relations) iff $|u_k|^2 + |v_k|^2 = 1$.

We now substitute these forms into the BCS Hamiltonian to find

$$\begin{aligned}\hat{H} = &\sum_k \xi_k \left[(|u_k|^2 - |v_k|^2)(\hat{\gamma}^\dagger_{k0}\hat{\gamma}_{k0} + \hat{\gamma}^\dagger_{k1}\hat{\gamma}_{k1} + 2|v_k|^2 + 2|v_k|^2 + 2u^*_k v^*_k \hat{\gamma}_{k1}\hat{\gamma}_{k0} + 2u_k v_k \hat{\gamma}^\dagger_{k1}\hat{\gamma}^\dagger_{k0}\right] \\ &+ \sum_k [(\Delta_k u_k v^*_k + \Delta^*_k u^*_k v_k)(\hat{\gamma}^\dagger_{k0}\hat{\gamma}_{k0} + \hat{\gamma}^\dagger_{k1}\hat{\gamma}_{k1} - 1) + \\ &+ (\Delta_k v^{*2}_k - \Delta^*_k u^{*2}_k)\hat{\gamma}_{k1}\hat{\gamma}_{k0} + (\Delta^*_k v^2_k - \Delta_k u^2_k)\hat{\gamma}^\dagger_{k0}\hat{\gamma}^\dagger_{k1} + \Delta_k F^*_k].\end{aligned} \tag{16}$$

This looks pretty complicated, but there is a way out. Let us try to find u_k and v_k such that the terms that are not diagonal would vanish. This means

$$2\xi_k u_k v_k + \Delta^*_k v^2_k - \Delta_k u^2_k = 0. \tag{17}$$

Together with the normalization condition $|u_k|^2 + |v_k|^2 = 1$, this results in the solutions

$$\begin{matrix} |u_k|^2 \\ |v_k|^2 \end{matrix} = \frac{1}{2}\left(1 \pm \frac{\xi_k}{E_k}\right). \tag{18}$$

Here E_k is the quasiparticle eigenenergy,

$$E_k = \sqrt{\xi_k^2 + |\Delta_k|^2}. \tag{19}$$

With these solutions, the Hamiltonian has the diagonal form

$$H = \sum_k E_k \left(\hat{\gamma}_{k0}^\dagger \hat{\gamma}_{k0} + \hat{\gamma}_{k1}^\dagger \hat{\gamma}_{k1}\right) + \sum_k \left(\xi_k - E_k + \Delta_k \langle \hat{c}_{k\uparrow}^\dagger \hat{c}_{-k\downarrow}\rangle^*\right). \tag{20}$$

It describes a system of free fermions (the quasiparticles) with spectrum E_k. The ground state has a rather interesting form,

$$|\mathrm{BCS}\rangle_{\mathrm{gnd}} = \prod_k \left[u_k + v_k \hat{c}_{k\uparrow}^\dagger \hat{c}_{-k\downarrow}^\dagger\right] |0\rangle. \tag{21}$$

Here $|0\rangle$ is the ground state for the $\hat{c}_{k\sigma}$ operators, that is

$$\hat{c}_{k\uparrow}|0\rangle = 0, \tag{22}$$

$$\hat{c}_{k\downarrow}|0\rangle = 0, \tag{23}$$

while $|\mathrm{BCS}\rangle_{\mathrm{gnd}}$ is the ground state for the operators $\hat{\gamma}_{k0}$, $\hat{\gamma}_{k1}$,

$$\hat{\gamma}_{k0}|\mathrm{BCS}\rangle_{\mathrm{gnd}} = 0, \tag{24}$$

$$\hat{\gamma}_{k1}|\mathrm{BCS}\rangle_{\mathrm{gnd}} = 0. \tag{25}$$

The last relation can be immediately checked by inverting Eqs. (12,13) thus obtaining

$$\hat{\gamma}_{k0} = u_k \hat{c}_{k\uparrow} - v_k \hat{c}_{-k\downarrow}^\dagger, \tag{26}$$

$$\hat{\gamma}_{k1}^\dagger = v_k^* \hat{c}_{k\uparrow} + u_k^* \hat{c}_{-k\downarrow}^\dagger. \tag{27}$$

III. DENSITY OF STATES FOR SUPERCONDUCTORS

Let us look into more detail into Eq. (19). Let us take Δ independent of k (this assumption can be justified by determining Δ self-consistently in the BCS theory). Then

$$E_k = \sqrt{\xi_k^2 + |\Delta|^2}. \tag{28}$$

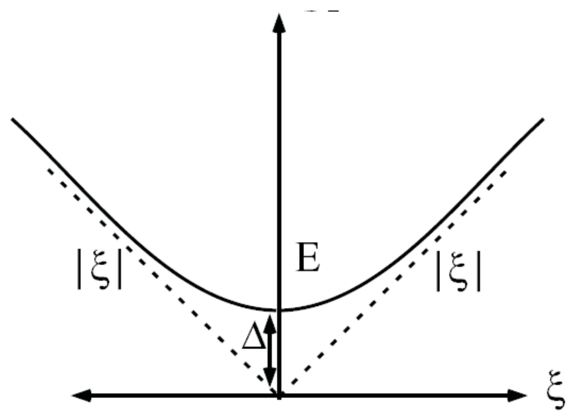


FIG. 1. Spectrum of quasiparticles in a BCS superconductor.

From here we can see that the minimum value of E_k is Δ: exciting quasiparticles in the superconductor is possible only of one provides an energy above this gap value. The spectrum of excitations of quasiparticles in a superconductor is shown in Fig. 1.

To find the density of states, we only have to make a change of variables in all the usual integrals over ξ_k. In other words

$$\mathcal{N}_{3D}^{(\mathrm{super})}(E)dE = \mathcal{N}_{3D}^{(\mathrm{norm})}(\xi)d\xi. \tag{29}$$

The density of states for the metal can be approximated with its value at the Fermi level (which corresponds to $\xi = 0$ with the notations here). Therefore we have

$$\mathcal{N}_{3D}^{(\mathrm{super})}(E) = \mathcal{N}_{3D}^{(\mathrm{norm})}(0)\frac{|E|}{\sqrt{E^2 - \Delta^2}}\Theta(|E| - \Delta). \tag{30}$$

The density of states in a superconductor is shown in Fig. (2).

This representation is called the **semiconductor picture** of supercondcutivity, due to its resemblance with the concept of semiconducting gap.

IV. TUNNELING IN JUNCTIONS WITH SUPERCONDUCTING LEADS

Consider now the following system, in which two superconductors are separated by an insulator that creates a tunnel barrier between them. Imagine that we try to measure the IV (current-voltage) characteristics of the system.

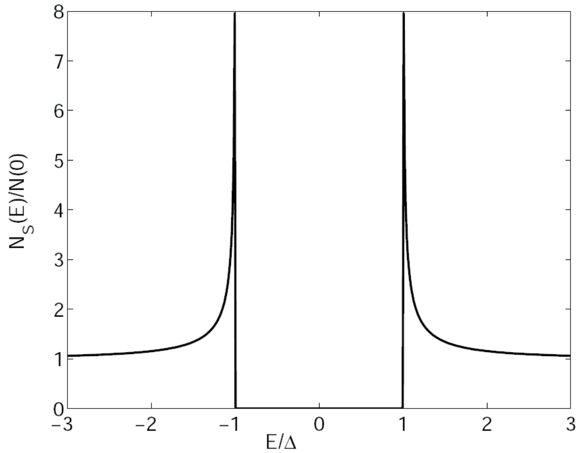
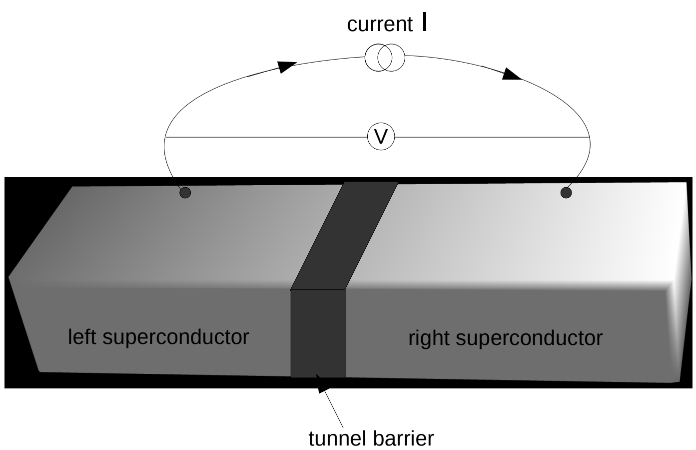


FIG. 2. Density of state in a superconductor. Figure from Ref. [1].



To calculate the current, it is natural to generalize the approach of the previous lecture to superconducting leads. The density of states has to be replaced by that of superconductors, as given by the BCS theory. It turns out that this density of states consists of a factor which is the same as for metals, which gets multiplied by a divergent part near the gap.

Summarizing all the discussion above, we can write the tunneling probability, including both the metallic and superconducting case, as

$$\Gamma_{1\to 2}(\delta E_{1\to 2}) = \frac{1}{e^2 R_{12}} \int_{-\infty}^{\infty} dE_1 \int_{-\infty}^{\infty} dE_2 \mathcal{N}_1(E_1)\mathcal{N}_2(E_2) f_1(E_1)[1 - f_2(E_2)]\delta(E_2 - E_1 - \delta E_{1\to 2}) \tag{31a}$$

$$= \frac{1}{e^2 R_{12}} \int_{-\infty}^{\infty} dE_1 \mathcal{N}_1(E_1)\mathcal{N}_2(E_1 + \delta E_{1\to 2}) f_1(E_1)[1 - f_2(E_1 + \delta E_{1\to 2})]\,, \tag{31b}$$

with $\mathcal{N}_{1,2}$ being the normalized density of states of the electrodes, as introduced above.

Furthermore, given the tunneling probabilities of Eq. (31), we can calculate the current through a single superconducting tunnel junction without charging effects

$$I_{1\to 2} = -e\left[\Gamma_{1\to 2}(\delta E_{1\to 2}) - \Gamma_{2\to 1}(\delta E_{2\to 1})\right] \tag{32}$$

$$= \frac{1}{eR_{12}}\left[\int_{-\infty}^{\infty} dE_1 \mathcal{N}_1(E_1)\mathcal{N}_2(E_1+eV) f_1(E_1)(1-f_2(E_1+eV))\right. \tag{33}$$

$$\left. - \int_{-\infty}^{\infty} dE_2 \mathcal{N}_1(E_2-eV)\mathcal{N}_2(eV) f_2(E_2)(1-f_2(E_2-eV))\right] \tag{34}$$

$$= \frac{1}{eR_{12}}\int_{-\infty}^{\infty} dE_1 \mathcal{N}_1(E_1)\mathcal{N}_2(E_1+eV)[f_1(E_1)-f_2(E_1+eV)], \tag{35}$$

which is a generalizations of the formulas we are used to from the previous lectures. To calculate the integrals a more convenient and at the same time physically justifiable approach is to introduce a broadening parameter into the density of states,

$$\mathcal{N}_{\text{Dynes}} = \left|\Re\left\{\frac{E+i\eta}{\sqrt{(E+i\eta)^2+\Delta^2}}\right\}\right| . \tag{36}$$

The parameter η is called Dynes parameter and it accounts for the finite life-time of quasi-particles in the superconductor. The symbol $\Re$ stands for the real part of the expression in curly brackets. This broadening of the density of states leads to nonzero currents flowing through the tunnel junction at voltages smaller than Δ/e (say for a NIS junction), which are often seen in experiments.

A. NIS junctions

For a single NIS junction, we just use a constant normal-metal density of states for $\mathcal{N}_1$ in Eq. (35) and a superconductor density of states for $\mathcal{N}_2$, see Fig. 3. At zero temperature, there is no current in the gap region (that is, for a bias $-\Delta < eV < \Delta$), simply because there are no states available there in the superconductor. At large values of the gap, the effect of the superconducting gap is not apparent anymore, and the sample behaves simply as a constant-conductivity element with resistance R_{12}.

In Fig. 4 we show the IV and dI/dV characteristics for a NIS junction.

SINIS devices The presentation below follows closely [5]. A superconductor-insulator-normal metal-insulator-superconducting (SINIS) device can be seen as an SET without gate, with superconducting leads and a normal metal island. SINIS are very useful devices, used as local coolers [2], secondary thermometers [3], and electron pumps [4]. Usually the charging

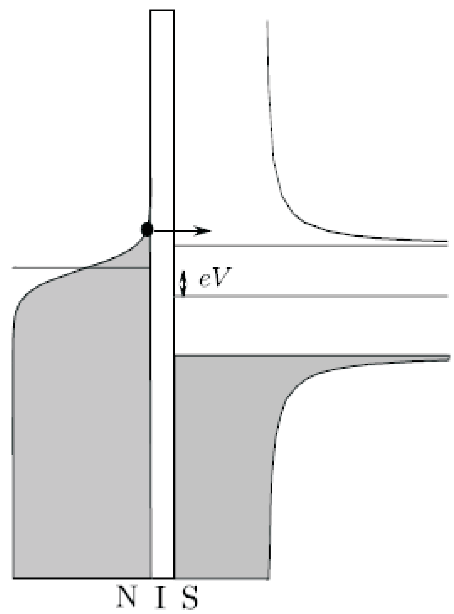


FIG. 3. Semiconductor model for NIS tunneling. Figure from [1].

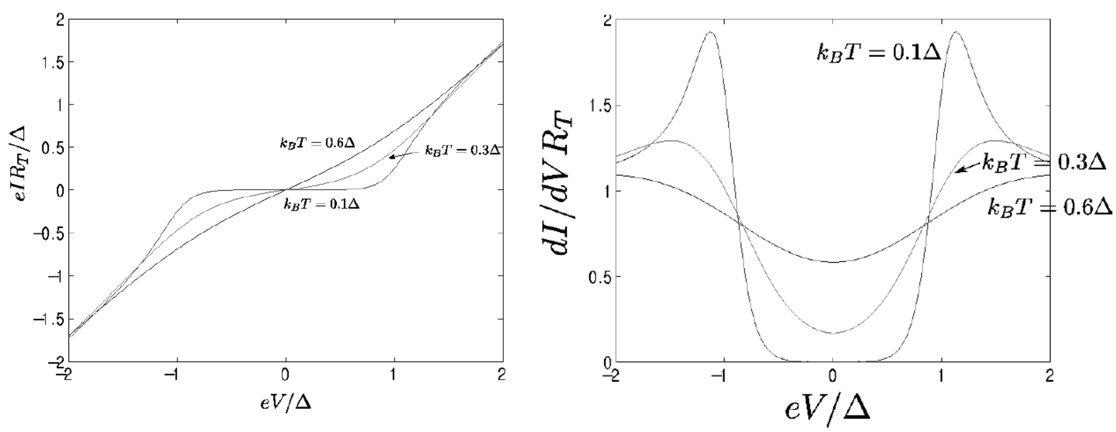


FIG. 4. IV and $\mathrm{d}I/\mathrm{d}V$ curves for a NIS junction at various temperatures. Figure from [1].

energy in a SINIS is so small that it can be neglected. However, in some applications, for instance as thermometers, the SINIS dimensions may become small enough to make charging effects become observable.

Figure 5 shows the I–V-curves and conductances of a SINIS structure at different operating temperatures for both zero and finite charging energy. Without charging energy (upper plots), the SINIS reproduces the I–V curve of a single NIS junction, with all features being

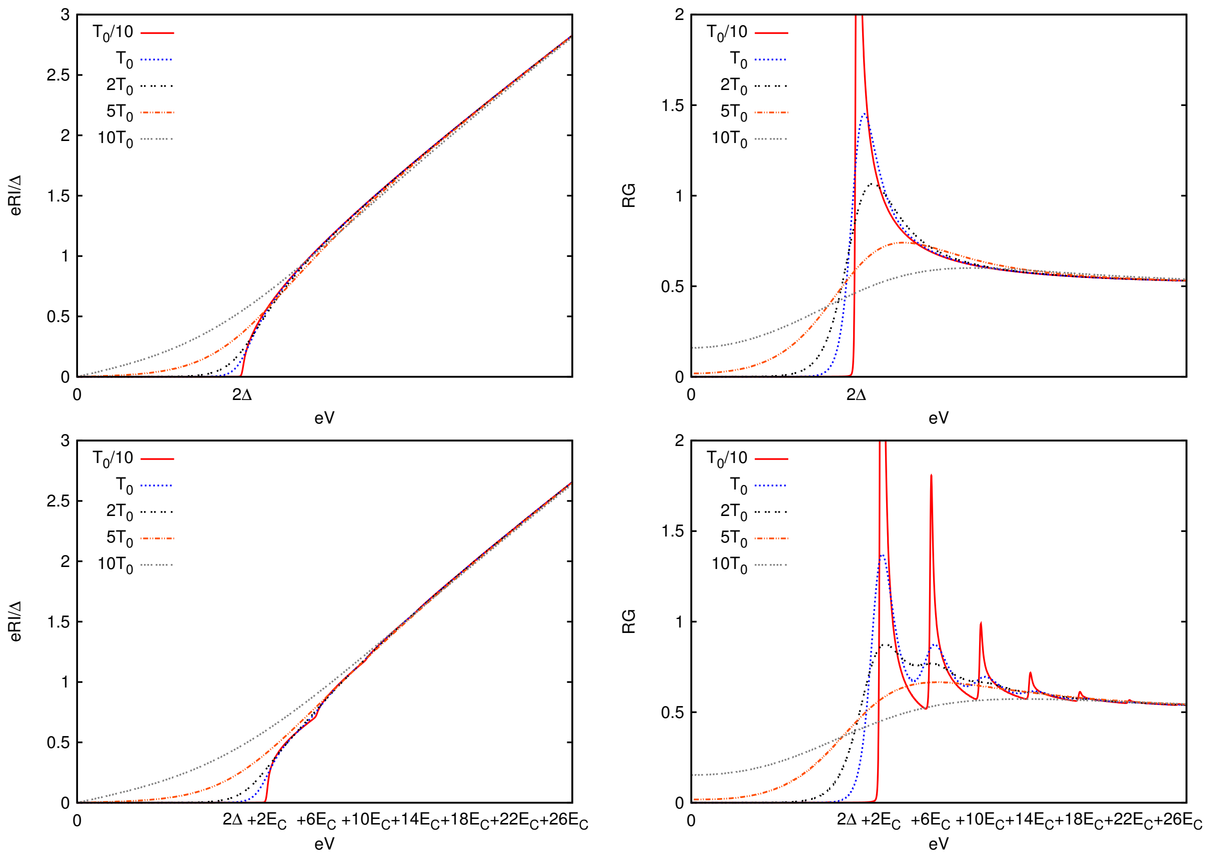


FIG. 5. Top: Current (left) and conductance (right) of a SINIS structure as function of applied voltage at different operating temperatures. The charging energy of the system is set to zero. Bottom: Current (left) and conductance (right) through a SINIS structure with finite charging energy ($E_C = 0.15\Delta$). The offset of the quasiparticle threshold is $2E_C$ and the periodicity of its repetition is $4E_C$. In all plots the temperature is $k_B T_0 = 3.92 \times 10^{-2}\Delta$. Figure and text from [5]

at twice the voltage, as we expect due to the fact that only half of the voltage applied to the SINIS drops over a single junction. With a finite charging energy (lower plots), the quasiparticle threshold is pushed up from $V = 2\Delta/e$ to $(2\Delta + 2E_C)/e$, and is repeated at $4E_C$ intervals with decreasing amplitude. This is visible especially well in the conductance plots in Fig. 5. At operating temperatures that are comparable to or larger than the charging energy, all features are smeared out. In Fig. 5 (for the case that $\Delta = 220$ μeV and thus $E_C = 33$ μeV), the curve with still barely visible oscillations conductance would correspond to a temperature of $2k_B T_0 \approx 17$ μeV, while in the curve at the next higher temperature of $5k_B T_0 \approx 43$ μeV the oscillations are smeared out entirely.

SINIS refrigerators

In Fig. 3 we show the main idea of NIS cooling and some experimental results are shown in Fig. 6. The existence of a superconducting gap selectively allows the hot electrons (at the tail of the Fermi distribution) to tunnel from the metal to the superconductor. Tunneling of the colder electrons is suppressed because there are no states available in the gap of the superconductor. As a result, the metal electrode cools.

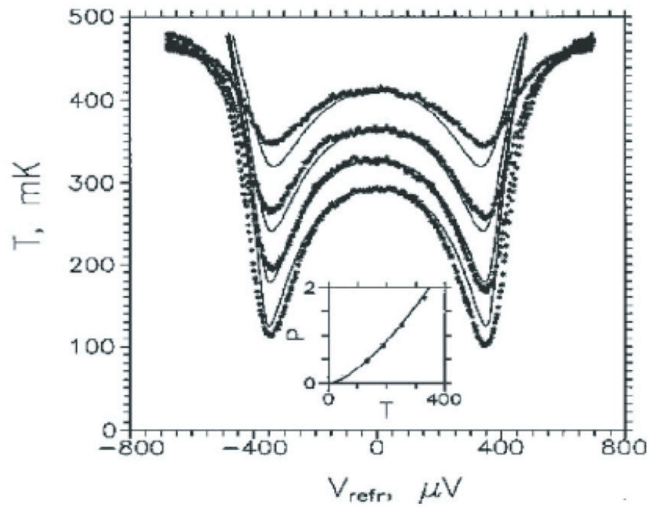


FIG. 6. Experimental results for cooling. Figure from [1].

In the case of two-junction structures such as SETs, SINIS, *etc.*, this results in the heating and cooling of the middle electrode (the island). This is due to the fact that, although the currents through the two junctions are the same, as it should be due to the conservation of charge (the charge of the island has to remain constant) the energy transfer (heat) does not need to satisfy such a conservation law.

It is straightforward to calculate the energy transferred per unit time out of the island (cooling power) as a function of the applied voltage,

$$P_{\mathrm{I}}(V) = \sum_{n=-\infty}^{\infty} p(n) \left[P_{\mathrm{I}\to\mathrm{L}}(n) - P_{\mathrm{L}\to\mathrm{I}}(n) + P_{\mathrm{I}\to\mathrm{R}}(n) - P_{\mathrm{R}\to\mathrm{I}}(n) \right] . \tag{37}$$

Here the powers $P_{\mathrm{L}\to\mathrm{I}}(n)$, $P_{\mathrm{I}\to\mathrm{L}}(n)$, $P_{\mathrm{R}\to\mathrm{I}}(n)$, and $P_{\mathrm{I}\to\mathrm{R}}(n)$ correspond to energy being transferred onto/off the island by tunneling into the left and right electrodes, at a given number of excess electrons n on the island. They can be calculated by multiplying the energy carried by each tunneling electron to the probability of tunneling per unit time (given by the

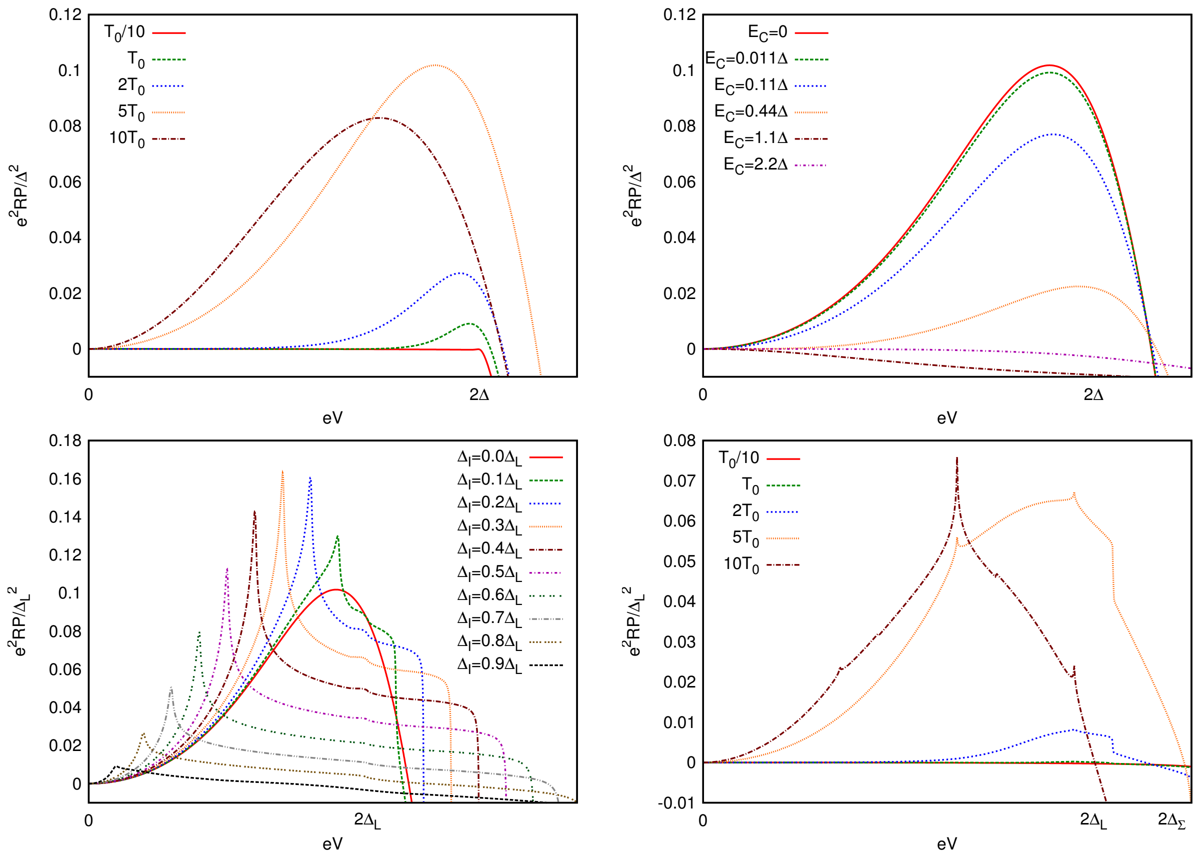


FIG. 7. Cooling powers in different double-island devices. Top left: SINIS structure with $E_C = 0$ at different temperatures. Top right: SINIS at fixed temperature $5T_0$ with different charging energies. Bottom left: SET with $5T_0$ and $E_C = 0$ for different gap sizes $\Delta_{\rm I} < \Delta_{\rm L}$ of the island. Bottom right: SET with $\Delta_{\rm I} = 0.2\Delta_{\rm L}$ and $E_C = 0.15\Delta_{\rm L}$ at different temperatures. In all plots $k_B T_0 = 3.92 \times 10^{-2}\Delta_{\rm L}$. Figure and text from [5].

density of states and Fermi factors), and then summing over energy states (see also Eqs. (31a), (31b)),

$$P_{\rm I\to X}(n) = \frac{1}{e^2 R_{IX}} \int_{-\infty}^{\infty} E E N_{\rm I}(E) N_{\rm X}[E + \delta E_{\rm I\to X}(n)] f_{\rm I}(E) \left[1 - f_{\rm X}(E + \delta E_{\rm I\to X}(n))\right]$$

$$P_{\rm X\to I}(n) = \frac{1}{e^2 R_{XI}} \int_{-\infty}^{\infty} E \left[E + \delta E_{\rm X\to I}(n)\right] N_{\rm I}\left[E + \delta E_{\rm X\to I}(n)\right] N_{\rm X}(E) \left[1 - f_{\rm I}(E + \delta E_{\rm X\to I}(n))\right] f_{\rm X}(E)\,,$$

where $\rm X = L, R$.

In Figure 7 we present a comparison between the cooling powers on the island Eq.(37) for SINIS (upper plots) and superconducting SET structures (lower plots). We observe that cooling exists only in an interval of bias voltages not much larger than the quasiparticle threshold at $2\Delta/e$ (we take identical superconducting leads with gap Δ). As seen from the left upper plot, as the temperature is lowered the process becomes much less efficient. The general effect of the charging energy is that it is detrimental to cooling (right plot, upper part). In the case of superconducting SET's (lower plots, Figure 7) the existence of singularity-matching peaks produces relatively sharp spikes in the cooling power. Essentially, the island is cooled down by BCS quasiparticles [6]. The left figure shows these features for the case of negligible charging energy (a SISIS structure with a relatively large middle electrode having a different gap than the leads). An interesting feature is also that the range of voltages over which cooling occurs is extended, due to the fact that now the quasiparticle threshold is at values $2\Delta_{\mathrm{L}} + 2\Delta_{\mathrm{I}}$. Finally, cooling in SET structures with finite charging energy is shown in the lower-right plot of Figure 7.

B. SIS junctions

For a single SIS junction, we will have to use a superconductor density of states for both $\mathcal{N}_1$ and $\mathcal{N}_2$ in Eq. (35). The semicornductor picture looks like in Fig. 8.

In the IVs, the most important difference that appears is that at zero temperature the current is zero in the voltage region $-\Delta_1 - \Delta_2 < eV < \Delta_1 + \Delta_2$. This is because in order for an electron to tunnel say from the superconductor 1 to superconductor 2, a Cooper pair has to be broken in superconductor 1 (and this costs a minimum energy of Δ_1) and then, when the electron arrives in superconductor 2 it will be an unpaired electron (which is a quasiparticle excitation that costs a minimum Δ_2).

V. THE JOSEPHSON EFFECT

But this is not everything. If you do the experiment, you'd see in the IV characteristics an electric current that flows across the junction even at zero voltage across the circuit! (kind of weird, isn't it? - a current flows without applying any bias voltage). This is the Josephson effect. It was predicted theoretically by Brian Josephson when he was a graduate student, and it ran against the accepted wisdom of that time (very famous scientists thought that, had this current existed, it should be a second-order effect in the tunneling, therefore very small and negligible). But Josephson put his money where the mouth is and did also

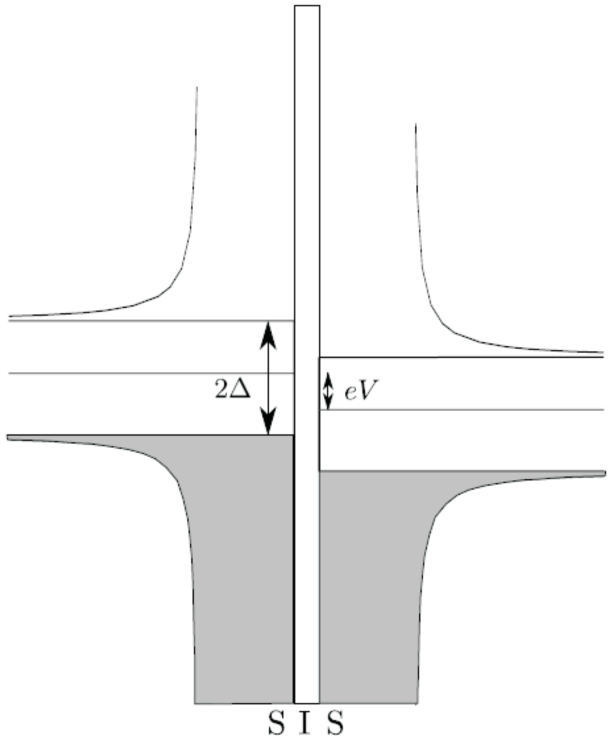
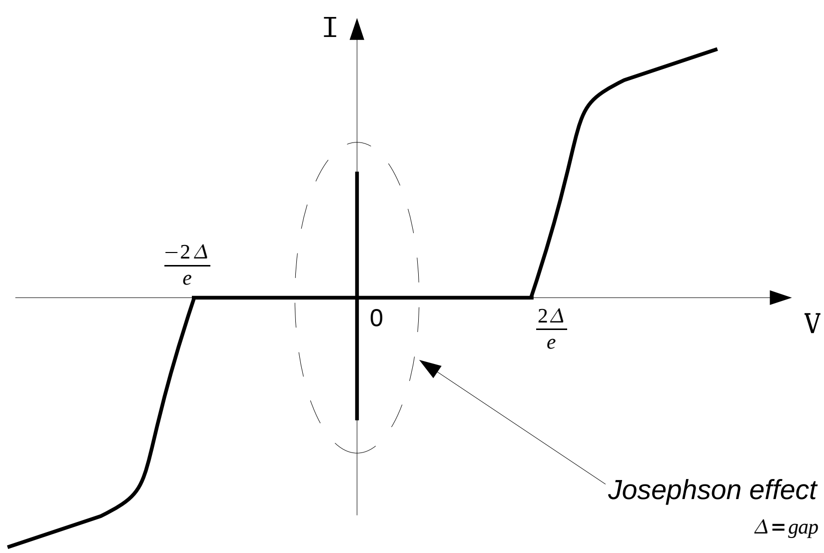


FIG. 8. Semiconductor model for SIS tunneling. Figure from [1].

the experiment!



Let us recall Eq. (21), the many-body BCS (Bardeen, Cooper, and Schrieffer) state

$$|\mathrm{BCS}\rangle = \prod_{k} \left(|u_k| + |v_k| e^{i\varphi_k} \hat{c}^{\dagger}_{k\uparrow} \hat{c}^{\dagger}_{-k\downarrow} \right) |0\rangle \,, \tag{39}$$

where $\hat{c}^{\dagger}_{k\uparrow} \hat{c}^{\dagger}_{-k\downarrow}$ shows that electrons are paired in Cooper pairs. In this expression we just put in evidence the fact that u_k and v_k are complex numbers; however it does not make sense to keep track of both of their phases, just of the phase difference between them, which we conveniently can take as the phase of v_k (an overall phase in front of this wavefunction does not matter). So $v_k = |v_k| \exp{(i\varphi_k)}$. Moreover, it can be argued that this phase should be the same for all Cooper pairs, so we can take $\varphi_k = \varphi$. From Eq. (17) we can see that this is also equivalent to operating with a complex superconducting order parameter $\Delta = |\Delta| \exp(i\varphi)$.

Next, let us try to extract some phenomenological information from it. So, what is the BCS function trying to tell us about how superconductors behave as circuit elements?

A. The Josephson voltage-phase relation

Suppose now that we apply a voltage on a superconductor (while keeping another one as reference). How do we write the new BCS many-body state?

Let us look at each electron. Assume that the single-electron Hamiltonian was $\hat{H}_0$, for which we can write a field equation (in direct analogy with the Schrödinger equation)

$$i\hbar \frac{\partial}{\partial t} \hat{\psi}\left(\mathbf{x}, t\right) = \hat{H}_0 \hat{\psi}\left(\mathbf{x}, t\right). \tag{40}$$

How do we introduce a voltage $V(t)$, which would produce an additional $-eV(t)$ in the Hamiltonian? We can simply preform the substitution

$$\hat{\psi}\left(\mathbf{x}, t\right) \rightarrow e^{-\frac{ie}{\hbar}\int_{-\infty}^{t} d\tau V(\tau)} \hat{\psi}\left(\mathbf{x}, t\right), \tag{41}$$

in the equation abovem which produces eactly what we want, namely $i\hbar\frac{\partial}{\partial t}\hat{\psi}\left(\mathbf{x}, t\right) = \hat{H}_0 \hat{\psi}\left(\mathbf{x}, t\right) \rightarrow i\hbar\frac{\partial}{\partial t}\hat{\psi}\left(\mathbf{x}, t\right) = \left[\hat{H}_0 - eV(t)\right] \hat{\psi}\left(\mathbf{x}, t\right).$

In terms of the annihilation and creation operators, this means

$$\hat{c}_k \rightarrow \hat{c}_k e^{-\frac{ie}{\hbar}\int_{-\infty}^{t} d\tau V(\tau)}, \tag{42}$$

$$\hat{c}^{\dagger}_k \rightarrow \hat{c}^{\dagger}_k e^{\frac{ie}{\hbar}\int_{-\infty}^{t} d\tau V(\tau)}. \tag{43}$$

so from Eq. (39) we have

$$|\mathrm{BCS}\rangle \rightarrow \prod_k \left[|u_k| + |v_k| e^{i\left(\varphi + \frac{2e}{\hbar}\int_{-\infty}^{t} d\tau V(\tau)\right)} \hat{c}_{k\uparrow}^{\dagger} \hat{c}_{-k\downarrow}^{\dagger}\right] |0\rangle\,, \tag{44}$$

where $\varphi(t) = \varphi + \frac{2e}{\hbar}\int_{-\infty}^{t} d\tau V(\tau)$ is the resulting new time-dependent BCS phase. Note that we assume here that the structure of the BCS many-body wavefunction does not change under the application of the voltage.

We would like now to obtain an equation of motion for this phase. By simple differentiation with respect to time we get the so-called **Josephson voltage-phase relation**:

$$\boxed{\frac{d\varphi(t)}{dt} = \frac{2e}{\hbar} V(t)} \tag{45}$$

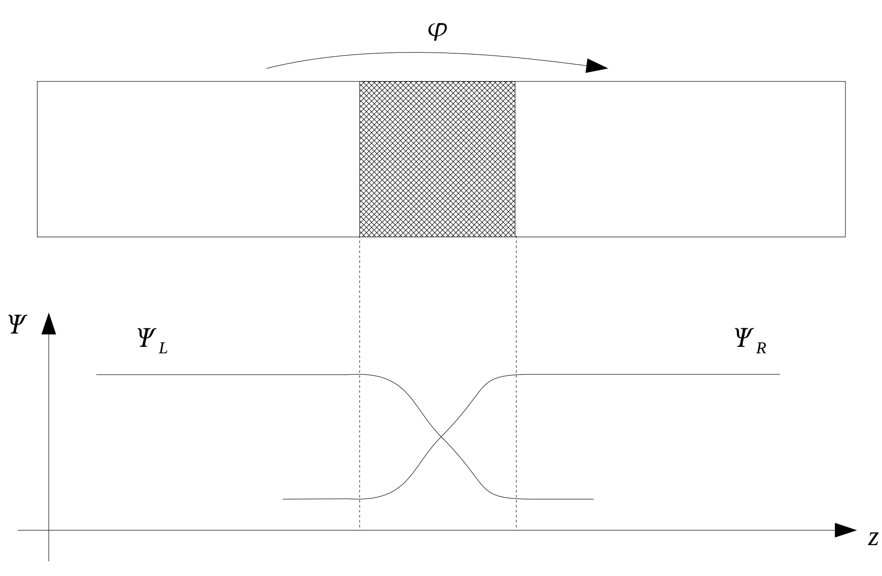
Note the factor $2e$ here: it is important to realize that it comes from the pairing of the electrons in a superconductor.

B. The Josephson current-phase relation

What we have established so far is that due to the special structure of the $|\mathrm{BCS}\rangle$ state, electrons pair up and they "phase-lock" to the same phase φ.

Consequence: think about one electron and its wavefunction. If it is in the right superconductor, its wavefunction will extend, via tunneling, into the left superconductor. If it is in the left well, the wavefunction will have a "tail" into the right superconductor.

In the same way, we attach the Cooper pair wavefunctions – call them Ψ_L, Ψ_R.



$\Rightarrow$

$\Psi \simeq \frac{1}{\sqrt{2}}\left[\Psi_L + \Psi_R e^{i\varphi}\right]$. You might object that this is not normalized, because already $\int |\Psi_L|^2 = 1$, $\int |\Psi_R|^2 = 1$; but we can think about it as an approximation, and in fact a good one if Ψ_R and Ψ_L do not overlap too much under the barrier.

Now note that the quantum-mechanical current of particles is

$$j \equiv \frac{i\hbar}{2m}\left[\left(\nabla\Psi^*\right)\Psi - \Psi^*\left(\nabla\Psi\right)\right] \simeq, \tag{46}$$

$$\simeq \frac{\hbar}{2m}\left[\Psi_L \frac{d}{dz}\Psi_R - \Psi_R \frac{d}{dz}\Psi_L\right] \cdot \sin\varphi \tag{47}$$

predicts a non-zero, and perhaps large current (... all Cooper pairs have the same phase φ so the tunneling currents of each will be added coherently and - if you imagine the electrons as paired - then there will be many of these Cooper pairs)

$$I_J = \text{const.} \times \sin\varphi, \tag{48}$$

where the constant depends on the tunneling matrix element between the two superconductors, as in the previous equation. The constant const. $= I_0$ is called the *critical current* of the junction.

Another useful quantity is

$$E_J = \frac{\Phi_0}{2\pi} I_0, \tag{49}$$

which is called the *Josephson energy.* Here $\Phi_0 = \frac{h}{2e} = 2 \cdot 10^{-7}$ Gcm^2 is referred to as the *flux quantum.*

Note: Above we have introduced the critical current of a junction in a somewhat phenomenological way. The expression for the critical current can be derived rigorously on the basis of the microscopic BCS theory, and it was first done in V. Ambegaokar and A.Baratoff, Phys. Rev. Lett. **10**, 486 (1963). The resulting Ambegaokar-Baratoff for the critical current of the junction reads

$$I_0 = \frac{\pi\Delta}{2eR_N} \tanh\frac{\Delta}{2k_B T}, \tag{50}$$

where Δ is the superconducting gap and R_N is the normal-state resistance.

For the purpose of this lecture, the essential result to remember is that $I_J = I_0 \cdot \sin\varphi$.

So, we found the so-called Josephson current-phase relation:

$$\boxed{I_J = I_0 \cdot \sin\varphi} \tag{51}$$

Although extremely simple, this relation is very powerful! Note that we can now already explain what is seen in tunneling experiments, namely the possibility of a non-zero current at zero bias voltage: from the Josephson voltage-phase relation at $V = 0$ we have $\varphi = \text{const.}$ therefore using the Josephson current-phase relation we get $\Rightarrow I_J = \text{const.}$ at $V = 0$.

VI. APPENDIX: NONUNIFORM SUPERCONDUCTORS: BOGOLIUBOV - DE GENNES EQUATIONS

In the derivation of the BCS theory we have assumed an infinite sample and uniform (zero) potential, thus we have used the periodic boundary conditions and the momentum state representation. However, the BCS theory can be formulated more generally (de Gennes), resulting in a nonuniform $|\Delta(\vec{r})|$.

The general Hamiltonian in second quantization is

$$H = \sum_\sigma \int d\vec{r}\psi_\sigma^\dagger(\vec{r})H_0(\vec{r})\psi_\sigma(\vec{r}) + \sum_{\sigma,\sigma'} \int d\vec{r} \int d\vec{r}' V_{\sigma,\sigma'}(\vec{r},\vec{r}')\psi_\sigma(\vec{r})^\dagger \psi_{\sigma'}(\vec{r}')^\dagger \psi_{\sigma'}(\vec{r}')\psi_\sigma(\vec{r}), \tag{52}$$

where

$$H_0(\vec{r}) = \frac{1}{2m}\left(-i\hbar\vec{\nabla} - e\vec{A}\right)^2 + U(\vec{r}) - \mu, \tag{53}$$

where in general $\vec{A}$ is the vector potential of a magnetic field. The interaction potential is local and corresponding to singlet spin coupling,

$$V_{\sigma,\sigma'}(\vec{r},\vec{r}') = V(\vec{r})\delta(\vec{r}-\vec{r}')\delta_{\sigma,\bar{\sigma}'}, \tag{54}$$

where $\bar{\sigma}$ is the opposite value ('negation of') σ. Now, using the momentum representation, one can easily check that the form of the Hamiltonians used above is reproduced.

If the system is not uniform, then the theory proceeds along the same lines, but u and v will be dependent on position.

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