MEC-E1050

FINITE ELEMENT METHOD IN SOLIDS 2021

Week 44-0

1 INTRODUCTION

LEARNING OUTCOMES

Students get an overall picture about prerequisites of the course, the roles of engineering models in structure modelling, and finite element method in displacement analysis of structures. The topics of week 44 are

- ◻ Structure modelling
- ◻ Prerequisites of MEC-E1050
- ◻ Engineering models
- ◻ Mathematica language and the finite element solver of MEC-E1050

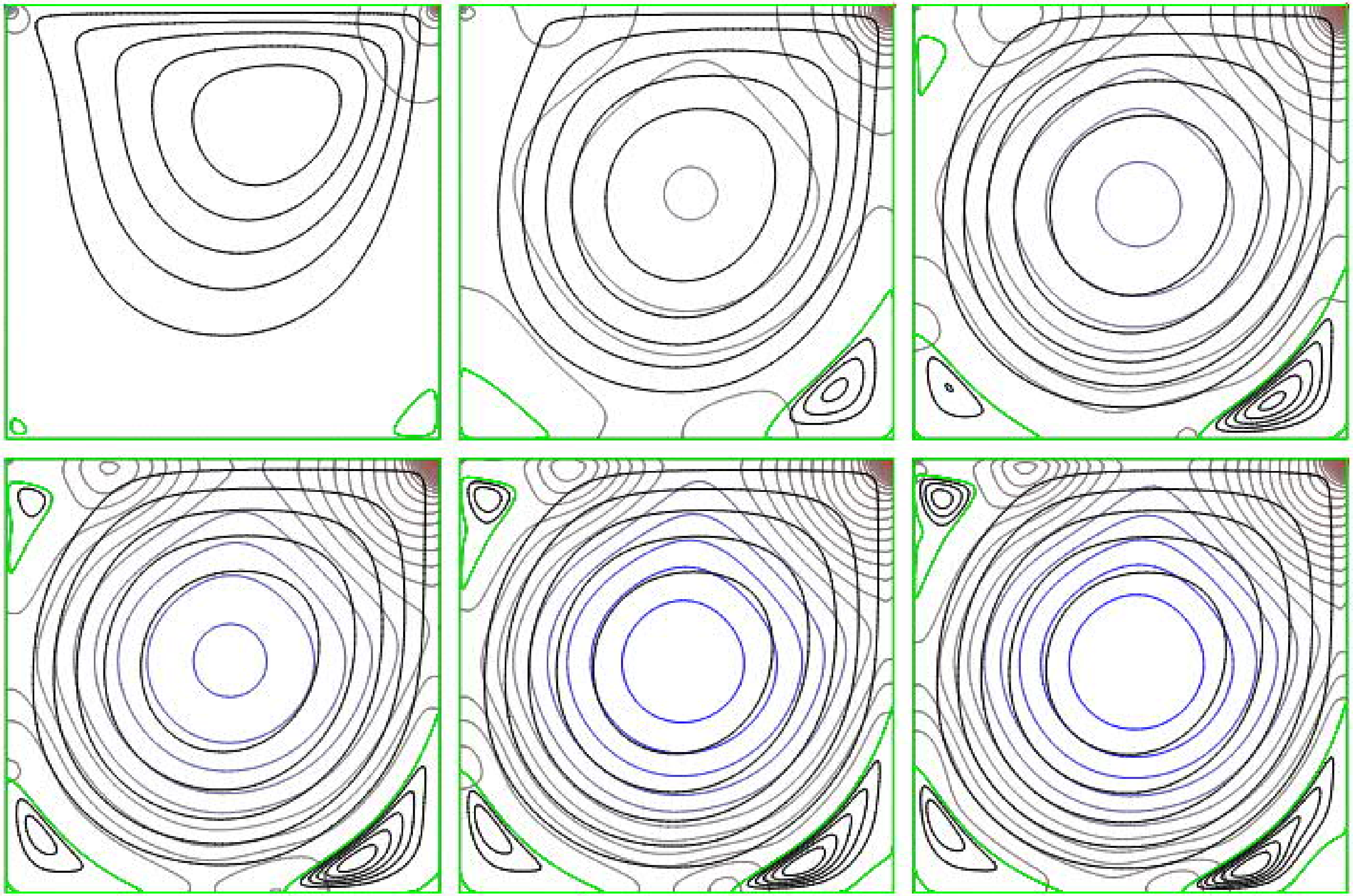
WHY FINITE ELEMENTS AND ITS THEORY?

Design of machines and structures: Solution to stress or displacement by analytical method is often impossible due to complex geometry, heterogeneous material etc. Lack of the "exact solution" to an "approximate problem" is not an issue in engineering work.

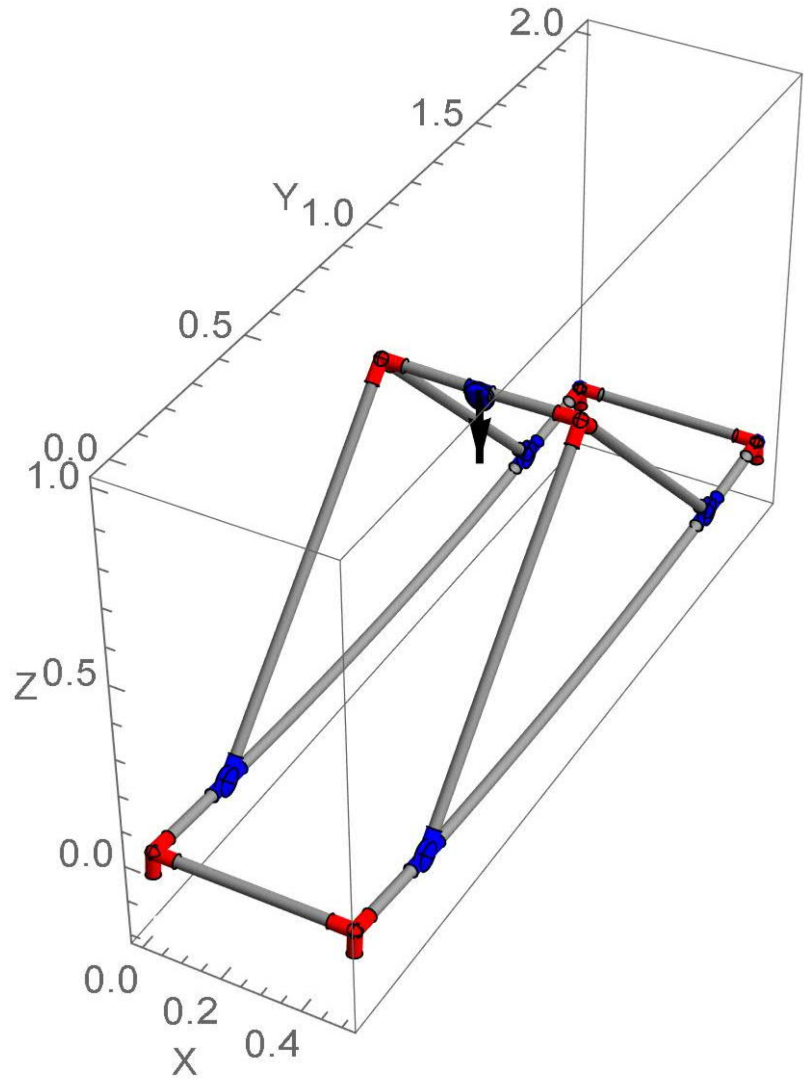
Finite element method is the standard of solid mechanics: Commercial codes in common use are based on the finite element method. A graphical user interface may make living easier, but a user should always understand what the problem is and in what sense it is solved!

Finite element method has a strong theory: Although approximate solution is acceptable, knowing nothing about the error is not acceptable.

FLUID MECHANICS APPLICATION

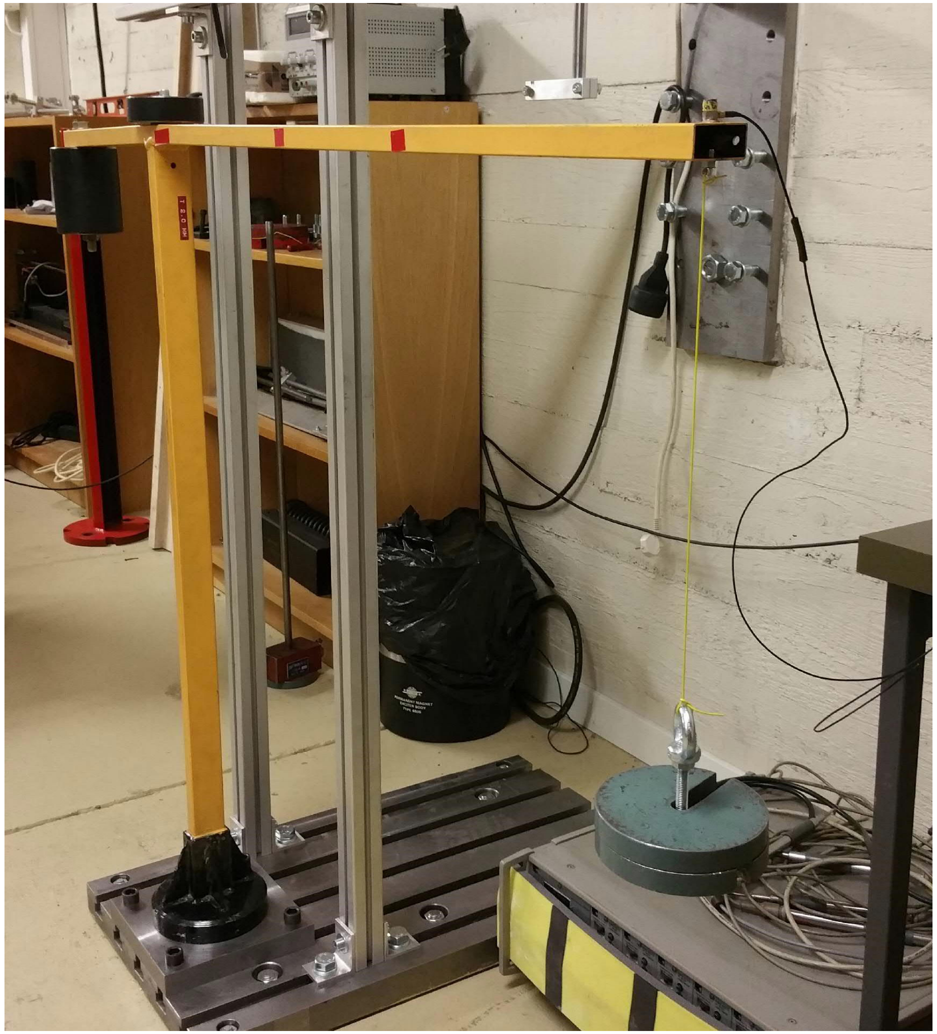


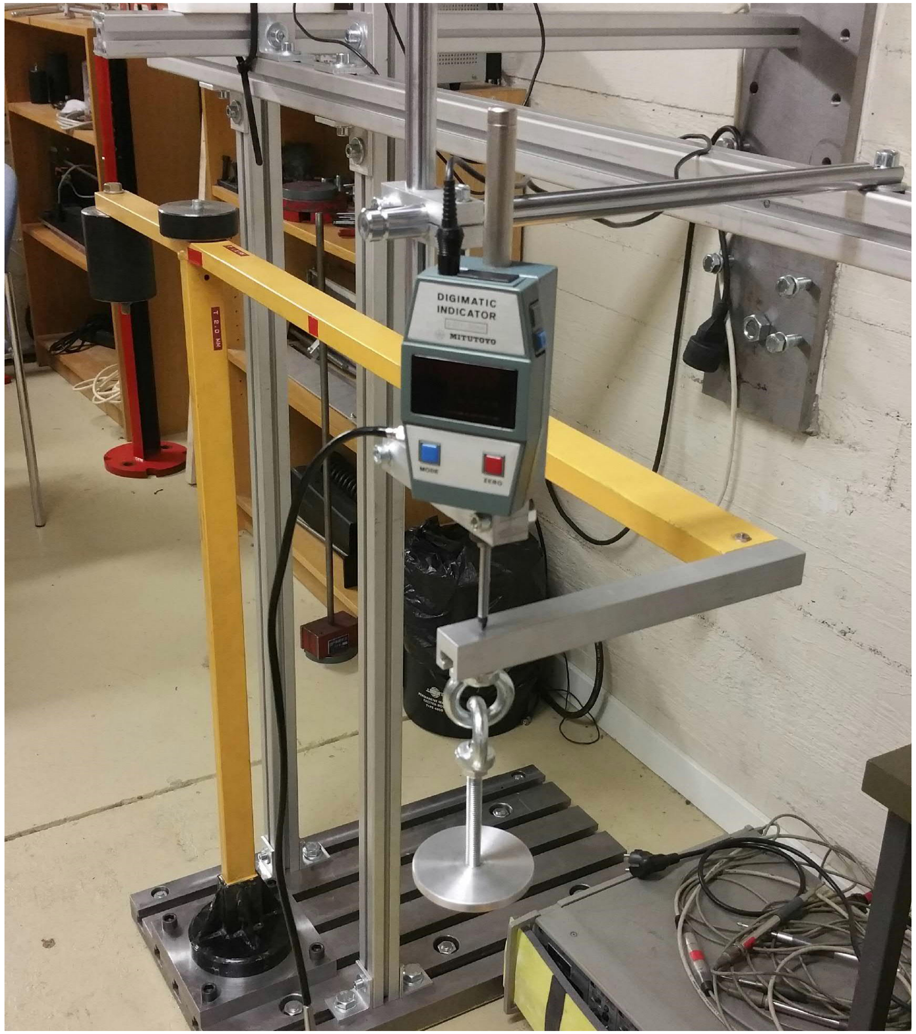
CONTINUUM MECHANICS APPLICATION



Week 44-5

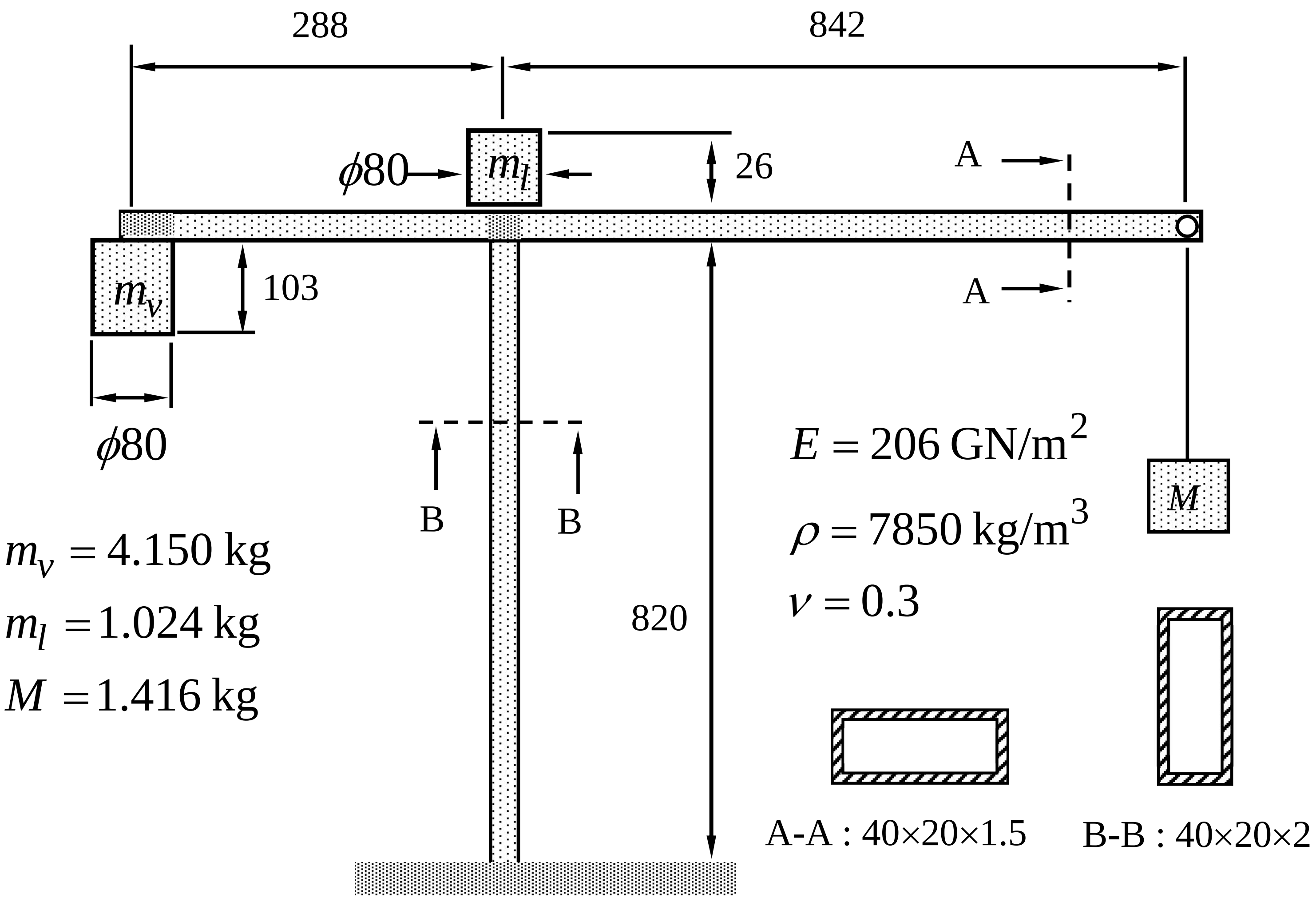
1.1 STRUCTURE MODELLING



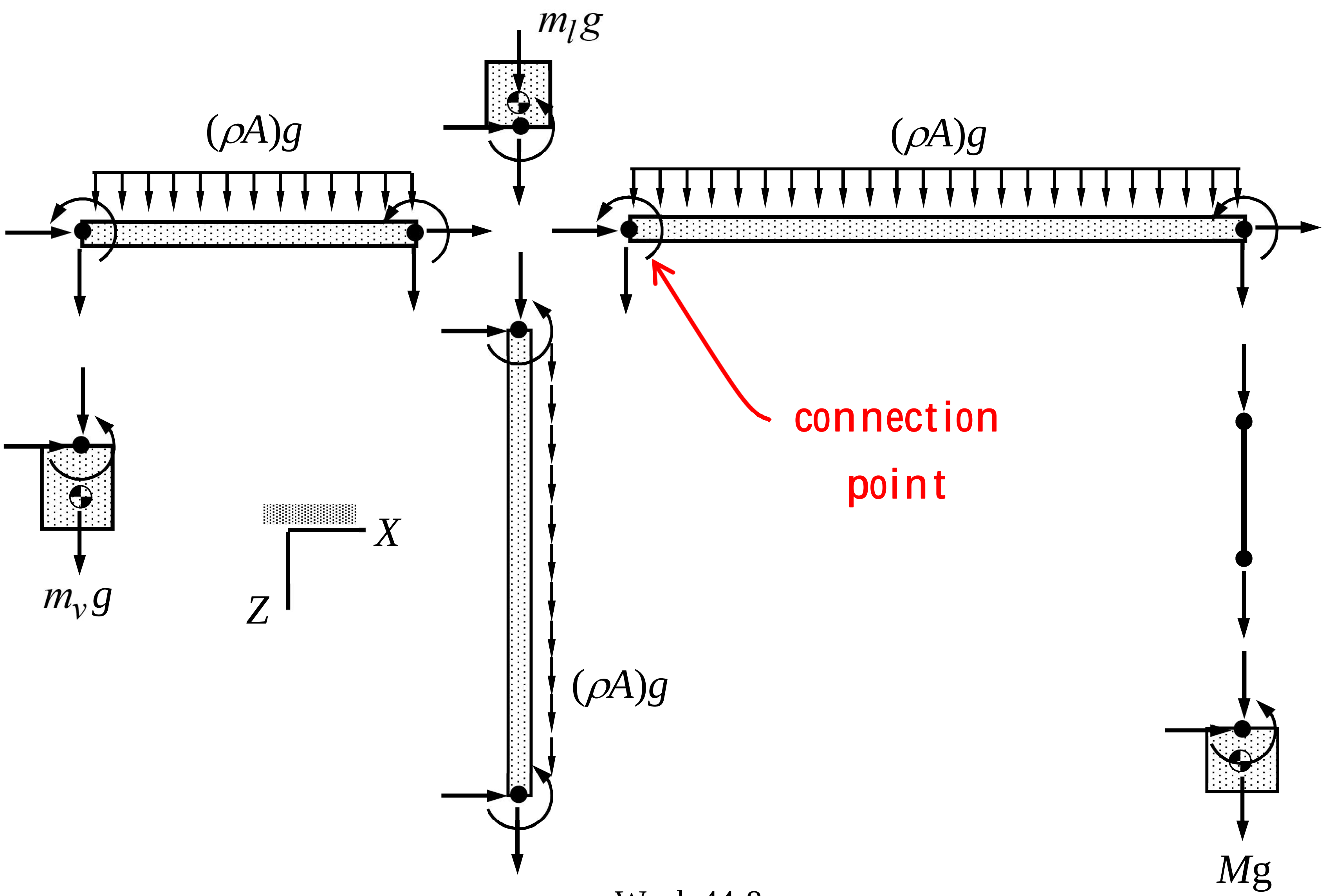


Week 44-6

IDEALIZED STRUCTURE

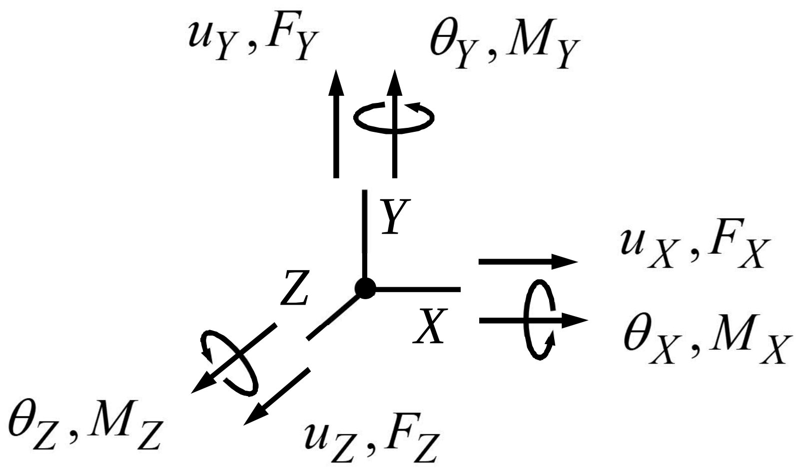


DIVIDE-AND-RULE



QUANTITIES OF ANALYSIS

The primary aim is to find displacements, rotations, forces and moments at the connection points of the structural parts. The components of the vector quantities (magnitude and direction) are taken to be positive in the directions of the coordinate axes.



Vector quantities are invariants in the sense $\vec{a} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k} = a_X\vec{I} + a_Y\vec{J} + a_Z\vec{K}$, and can be transformed from one coordinate system to another using the property.

MODELLING STEPS

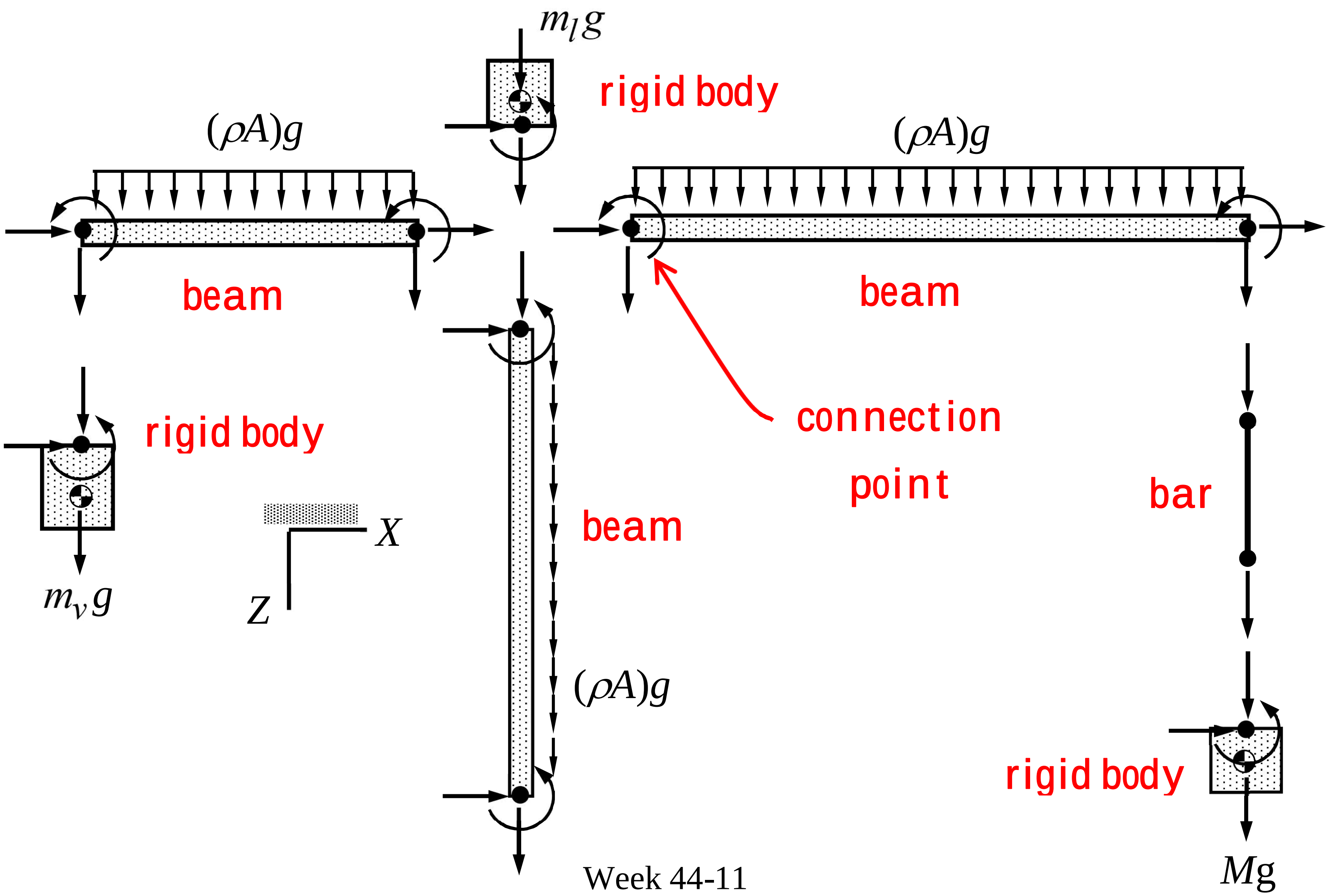
- **Crop**: Decide the boundary of a structure. Interaction with surroundings need to be described in terms of known forces, moments, displacements, and rotations. All uncertainties with this respect bring uncertainty to the model too.

- **Idealize**: Simplify the geometry. Ignoring the details not likely to affect the outcome may simplify the analysis a lot.

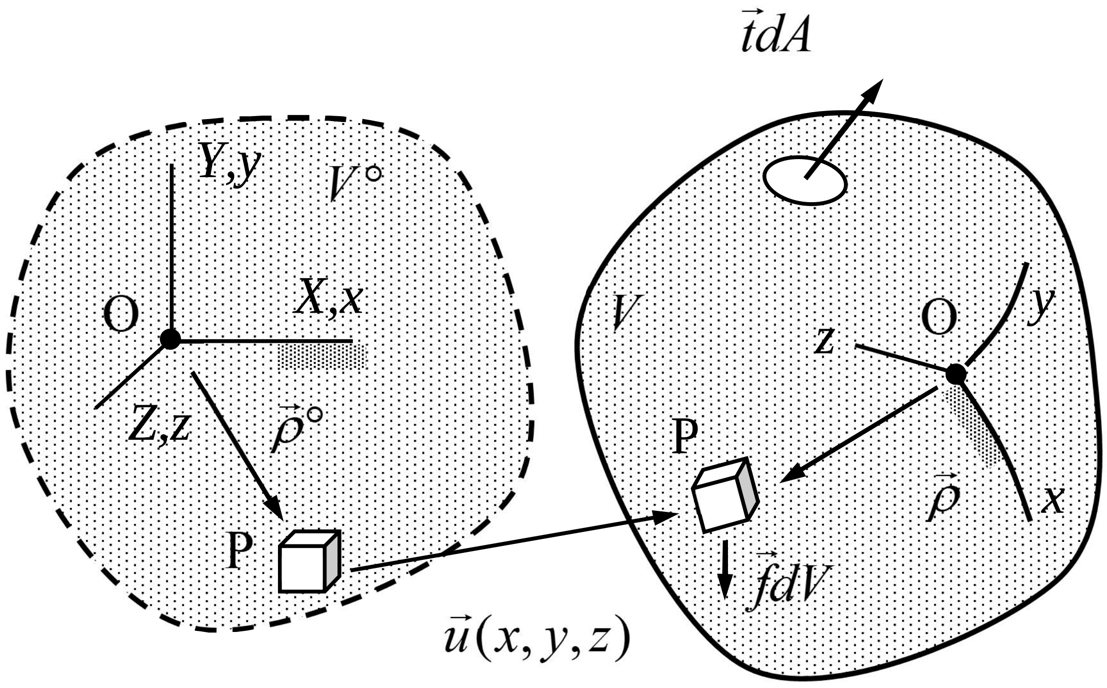
- **Parameterize**: Assign symbols to geometric and material parameter of the idealized structure. Measure or find the values needed in numerical calculations.

- **Divide-and-rule**: Represent a complex structure as a set of structural parts interacting through connection points.

1.2 ENGINEERING MODELS



SOLID MODEL



The primary unknowns are $u(x,y,z)$, $v(x,y,z)$, $w(x,y,z)$, ($\phi(x,y,z)$, $\theta(x,y,z)$, $\psi(x,y,z)$). Material elements may translate, rotate, and deform. In short, for points P and Q of an element $\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta}_{\mathrm{P}} \times \vec{\rho}_{\mathrm{PQ}} + \vec{\rho}_{\mathrm{PQ}} \cdot \vec{\vec{\varepsilon}}_{\mathrm{P}}$. Displacement follows from stress-strain relationship (generalized Hooke's law) and equilibrium of material elements.

- Let us consider the displacement of a small material element centered at point P. As the material element is assumed to be small, first two terms of the Taylor series represent the displacement inside the material element

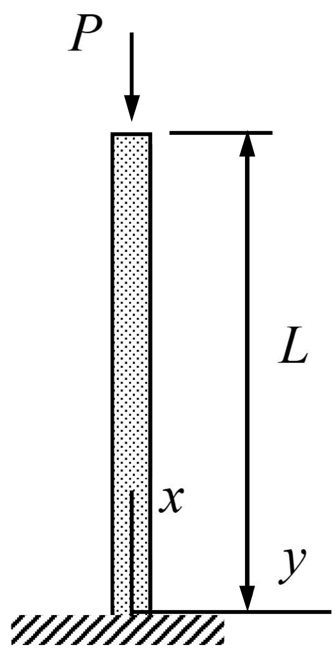
$$\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\rho}_{\mathrm{PQ}} \cdot (\nabla \vec{u})_{\mathrm{P}},$$

where the relative position vector $\vec{\rho}_{\mathrm{PQ}} = \vec{r}_{\mathrm{Q}} - \vec{r}_{\mathrm{P}}$. Division of the displacement gradient into its anti-symmetric and symmetric parts $(\nabla \vec{u})_{\mathrm{P}} = \overleftrightarrow{\theta}_{\mathrm{P}} + \overleftrightarrow{\varepsilon}_{\mathrm{P}}$ and using the concept of an associated vector $\vec{\theta}$ to an antisymmetric tensor $\overleftrightarrow{\theta}$, gives

$$\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta}_{\mathrm{P}} \times \vec{\rho}_{\mathrm{PQ}} + \vec{\rho}_{\mathrm{PQ}} \cdot \overleftrightarrow{\varepsilon}_{\mathrm{P}}.$$

The terms describe effects of translation, small rigid body rotation, and deformation (shape distortion) when the rotation part is small. Stress acting on the material element depends only on strain $\overleftrightarrow{\varepsilon}_{\mathrm{P}}$.

EXAMPLE. The cross section of a cylindrical body is square of side length h. Density ρ, Young's modulus E, and Poisson's ratio ν of the linearly elastic isotropic and homogeneous material are constants. The body is loaded by a constant traction of magnitude P/h^2 at its free end. Determine stress $\vec{\vec{\sigma}}$ and displacement $\vec{u}$ using the solid model. Assume that the transverse (to the axis) displacement is not constrained by the support.



Answer $u = -\dfrac{P}{Eh^2}x$, $v = \nu\dfrac{P}{Eh^2}y$, $w = \nu\dfrac{P}{Eh^2}z$

$$\sigma_{xx} = -\frac{P}{h^2}\ , \ \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$$

- The component forms of the equilibrium equations and constitutive equations of a linearly elastic isotropic material in a Cartesian $(x, y, z)-$coordinate system are

$$\begin{Bmatrix} \partial\sigma_{xx}/\partial x + \partial\sigma_{yx}/\partial y + \partial\sigma_{zx}/\partial z + f_x \\ \partial\sigma_{xy}/\partial x + \partial\sigma_{yy}/\partial y + \partial\sigma_{zy}/\partial z + f_y \\ \partial\sigma_{xz}/\partial x + \partial\sigma_{yz}/\partial y + \partial\sigma_{zz}/\partial z + f_z \end{Bmatrix} = 0,$$

$$\begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial w/\partial z \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix}, \text{ and } \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix} = \begin{Bmatrix} \sigma_{yx} \\ \sigma_{zy} \\ \sigma_{xz} \end{Bmatrix} = G\begin{Bmatrix} \partial u/\partial y + \partial v/\partial x \\ \partial v/\partial z + \partial w/\partial y \\ \partial w/\partial x + \partial u/\partial z \end{Bmatrix}.$$

- Let us assume that the non-zero stress and displacement components are $\sigma_{xx}(x)$, $u(x)$, $v(y)$ and $w(z)$. The axial stress follows from the equilibrium equation and the known traction at the free end $x = L$:

$$\frac{d\sigma_{xx}}{dx} = 0 \quad 0 < x < L \quad \text{and} \quad \sigma_{xx}(L) = -\frac{P}{h^2} \quad \Rightarrow \quad \sigma_{xx}(x) = -\frac{P}{h^2}.$$

- Generalized Hooke's law written for the uniaxial stress implies that

$$\frac{du}{dx} = \frac{\sigma_{xx}}{E} = -\frac{P}{Eh^2}, \quad \frac{dv}{dy} = -\frac{\nu}{E}\sigma_{xx} = \nu\frac{P}{Eh^2}, \quad \frac{dw}{dz} = -\frac{\nu}{E}\sigma_{xx} = \nu\frac{P}{Eh^2}.$$

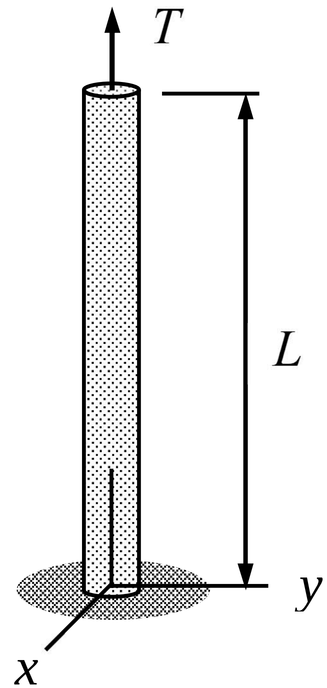
Axial displacement vanishes at the support and the transverse displacement at the axis:

$$\frac{du}{dx} = -\frac{P}{Eh^2} \quad 0 < x < L \quad \text{and} \quad u(0) = 0 \quad \Rightarrow \quad u(x) = -\frac{P}{Eh^2}x,$$ ←

$$\frac{dv}{dy} = \nu\frac{P}{Eh^2} \quad -\frac{1}{2}h < y < \frac{1}{2}h \quad \text{and} \quad v(0) = 0 \quad \Rightarrow \quad v(y) = \nu\frac{P}{Eh^2}y,$$ ←

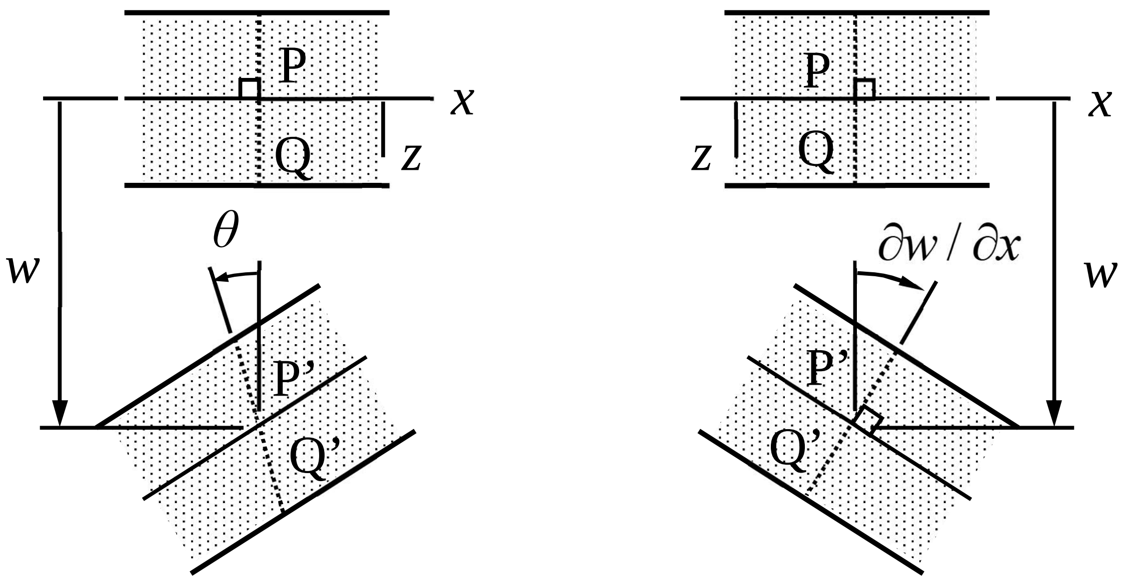
$$\frac{dw}{dz} = -\nu\frac{P}{Eh^2} \quad -\frac{1}{2}h < z < \frac{1}{2}h \quad \text{and} \quad w(0) = 0 \quad \Rightarrow \quad w(z) = \nu\frac{P}{Eh^2}z.$$ ←

EXAMPLE. Consider a torsion of a cylindrical body of length L and circular cross-section of radius R. Shear modulus G of the material is constant. If one end is fixed and the other end is free to rotate, determine the relationship between torque T and rotation angle ψ at the free end. Assume that $u = -\psi(z)y$, $v = \psi(z)x$, and $w = 0$.



Answer $T = \dfrac{I_{rr}G}{L}\psi$ where $I_{rr} = \dfrac{\pi}{2}R^4$.

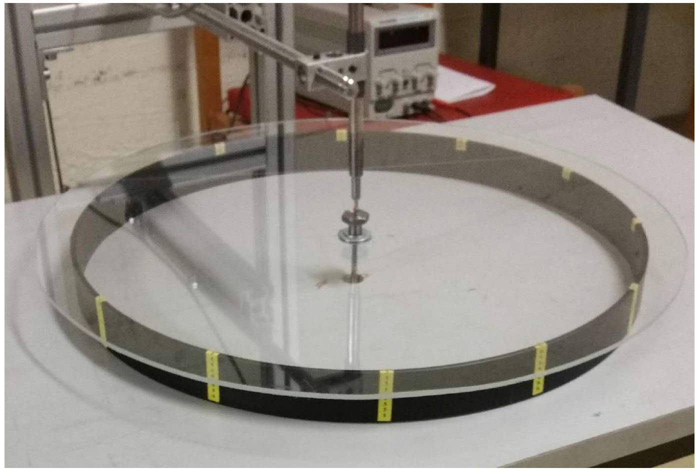
PLATE MODEL

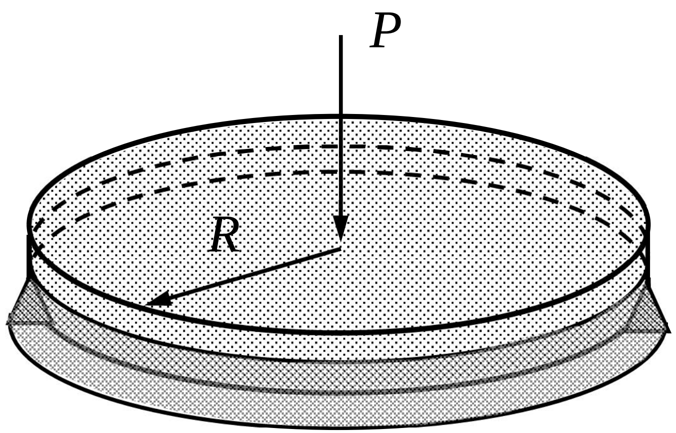


Reissner-Mindlin **Kirchhoff**

The primary unknowns are $u(x,y)$, $v(x,y)$, $w(x,y)$, $\phi(x,y)$, $\theta(x,y)$, $\psi(x,y)$. Line segments perpendicular to the mid/reference-plane remain straight in deformation (Reissner-Mindlin) and perpendicular to the mid-plane (Kirchhoff). Mathematically $\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta}_{\mathrm{P}} \times \vec{\rho}_{\mathrm{PQ}}$. Normal stress σ_{zz} is negligible.

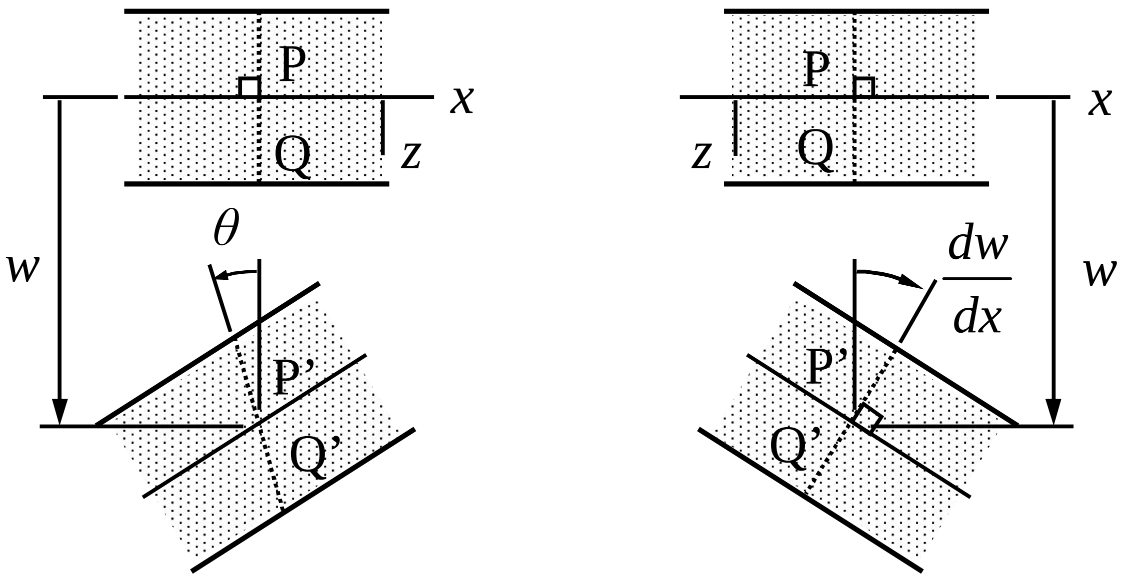
EXAMPLE. A simply supported circular body of radius R and thickness t is loaded by a point force P acting at the midpoint as shown in the figure. Determine the transverse displacement w at the midpoint by using the plate model. Young's modulus E and Poisson's ratio ν of the isotropic material are constants. Assume that displacement depends on the radial coordinate only.





Answer: $$w(0) = -\frac{1}{16\pi}\frac{PR^2}{D}\frac{3+\nu}{1+\nu} = -\frac{3}{4\pi}\frac{PR^2}{Et^3}(3+\nu)(1-\nu)$$

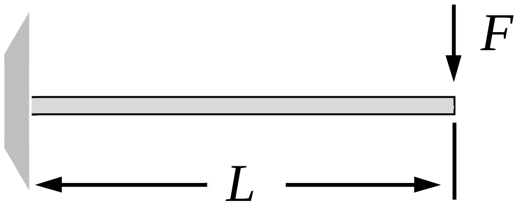
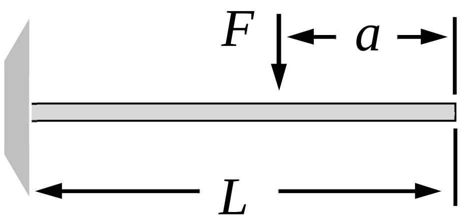
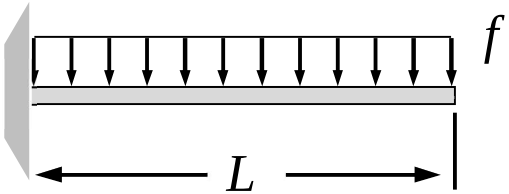
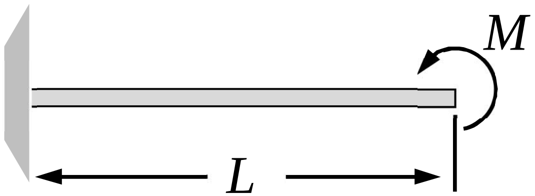
BEAM MODEL



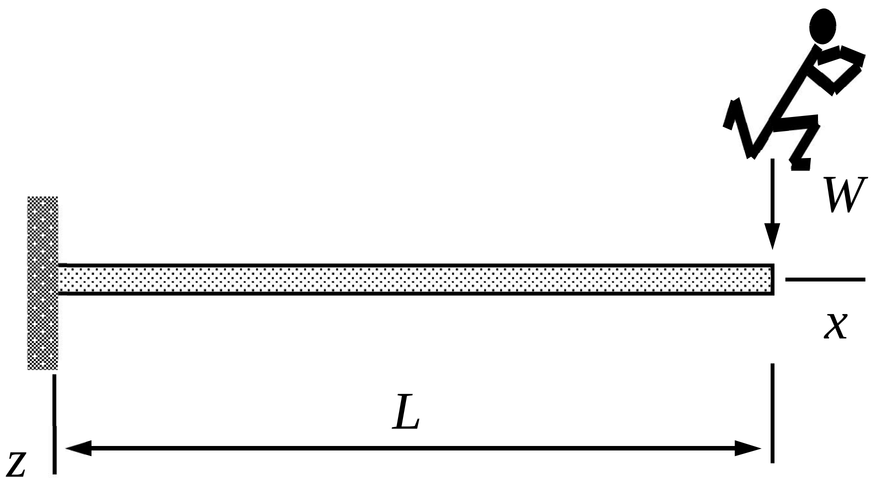
Timoshenko **Bernoulli**

The primary unknowns are $u(x)$, $v(x)$, $w(x)$, $\phi(x)$, $\theta(x)$, $\psi(x)$. Normal planes to the (material) axis of beam remain planes (Timoshenko) and normal to the axis (Bernoulli) in deformation. Mathematically $\vec{u}_{\text{Q}} = \vec{u}_{\text{P}} + \vec{\theta}_{\text{P}} \times \vec{\rho}_{\text{PQ}}$. Transverse normal stress is negligible i.e. $\sigma_{yy} = \sigma_{zz} \ll \sigma_{xx}$.

BEAM BENDING

Loading case	Deflection (tip)	Rotation (tip)
	$w = \frac{FL^3}{3EI}$	$\theta = -\frac{dw}{dx} = -\frac{FL^2}{2EI}$
	$w = \frac{F(a-L)^2(a+2L)}{6EI}$	$\theta = -\frac{dw}{dx} = -\frac{F(L-a)^2}{2EI}$
	$w = \frac{f L^4}{8EI}$	$\theta = -\frac{dw}{dx} = -\frac{f L^3}{6EI}$
	$w = -\frac{L^2 M}{2EI}$	$\theta = -\frac{dw}{dx} = \frac{LM}{EI}$

EXAMPLE. A rigidly supported springboard of length L and cross-sectional area $A = bh$ is levelled without loading. Under which conditions displacement at the free end and stress at the support do not exceed the limit values δ and σ_{cr}, respectively, if a person of weight W is standing at the free end? Use the beam model and assume that the stress and the transverse displacement are related by $\sigma_{xx} = -Ezd^2w/dx^2$.

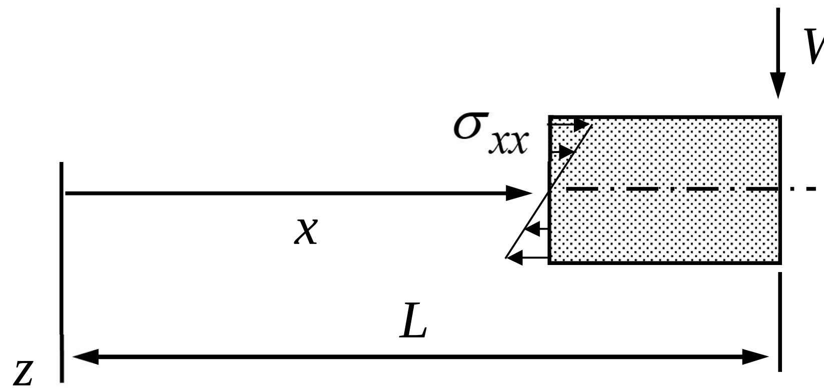


Answer $4\dfrac{WL^3}{Ebh^3} \le \delta$ and $6\dfrac{WL}{bh^2} \le \sigma_{\text{cr}}$

- The relationship between the axial stress and transverse displacement follows from Hooke's law and assumptions of the beam model. Moment equilibrium gives

$$-\int_{-h/2}^{h/2} z\sigma_{xx} b\,dz - W(L-x) = 0 \quad \Rightarrow$$

$$E\frac{bh^3}{12}\frac{d^2w}{dx^2} - W(L-x) = 0$$

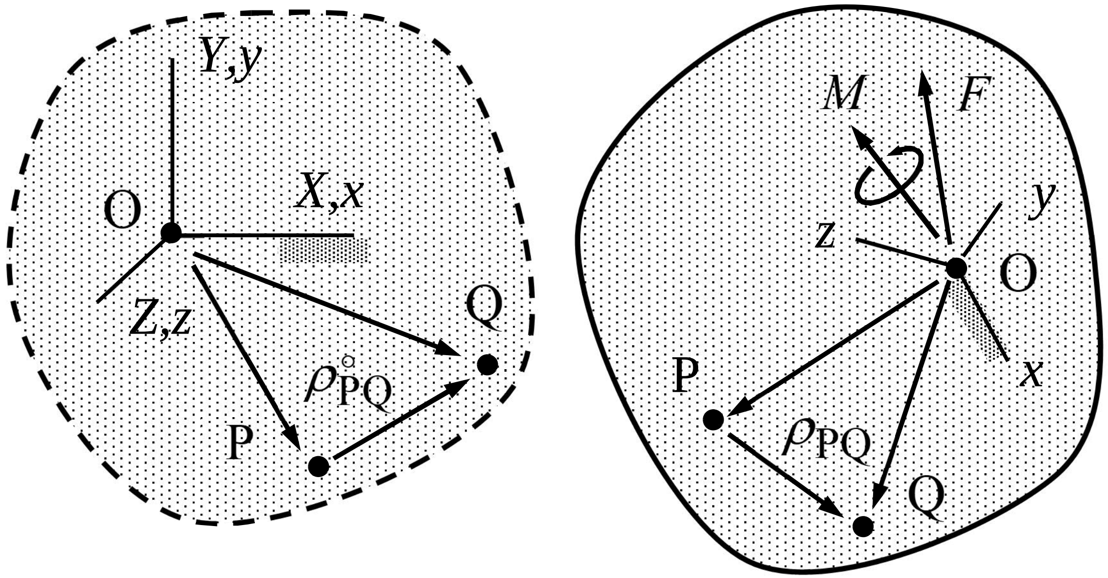


- Transverse displacement and its derivative vanish at the support. Hence

$$\frac{d^2w}{dx^2} = 12\frac{W}{Ebh^3}(L-x) \quad \Rightarrow \quad w(x) = 2\frac{W}{Ebh^3}x^2(3L-x).$$

- Therefore $w(L) = 4\frac{WL^3}{Ebh^3} \le \delta$ and $\sigma_{\max}(0) = 6\frac{WL}{bh^2} \le \sigma_{\mathrm{cr}}$.

RIGID BODY



The primary unknowns are u, v, w, ϕ, θ, ψ of the translation point. Body may translate and rotate but distance between any two points P and Q is constant. Mathematically, e.g., $\vec{u}_{\mathrm{Q}} = \vec{u}_{\mathrm{P}} + \vec{\theta}_{\mathrm{P}} \times \vec{\rho}_{\mathrm{PQ}}$. Rigid body idealization is useful when rigidities of structural parts differ significantly. Forces acting on the body are represented by a force-moment pair.

1.3 DISPLACEMENT ANALYSIS

- Idealize a complex structure as a set of structural parts, whose behavior can be approximated by using the usual engineering models (bar, beam, plate, rigid body etc.).

- Write down the equilibrium equations at the connections (Newton **III**), the force-displacement relationships of the structural parts, and constraints concerning the nodal displacements (displacements and rotations should match).

- Solve the nodal displacements and rotations and the forces and moments acting on the structural parts (elements in FEM) from the equation system.

- Determine the stress in the structural parts one-by-one according to the engineering model used (optional step).

NEWTON's LAWS OF MOTION

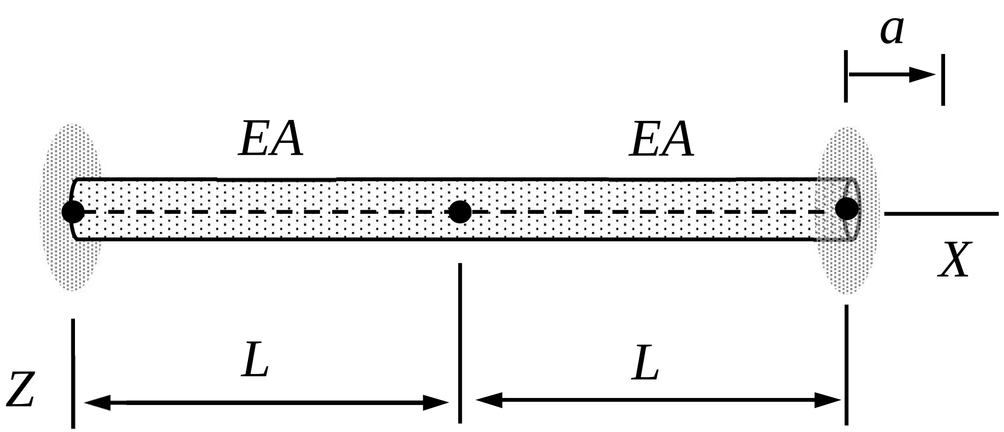
I In an inertial frame of reference, an object either remains at rest or continues to move at a constant velocity, unless acted upon by a force.

II The vector sum of the forces on an object is equal to the mass of that object multiplied by the acceleration of the object (assuming that the mass is constant).

III When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body.

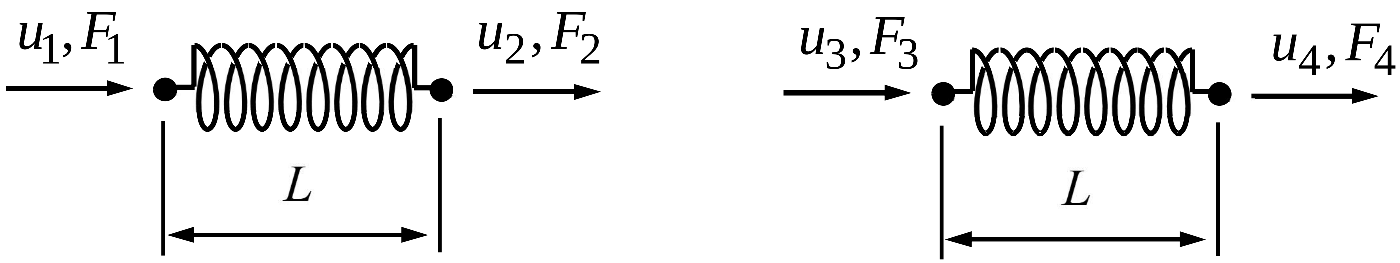
Newton's laws in their original forms apply to particles only. The formulation for rigid bodies and deformable bodies requires slight modifications.

EXAMPLE. A connector bar is welded at its ends to rigid walls. If the right end wall displacement is a, determine the displacements of connection points 1, 2, and 3 and the forces acting on structural parts. Cross sectional area A and Young's modulus of the material E are constants and the displacement force relationship of a bar is the same as that of a spring with coefficient $k = EA / L$. Model the structure as a collection of two bars (1 and 2).



Answer $u_1 = 0,\ u_2 = u_3 = \frac{1}{2}a,\ u_4 = a,\ F_1 = F_3 = -\frac{1}{2}ka,\ F_2 = F_4 = \frac{1}{2}ka.$

- As the structural parts can be considered as springs of coefficient $k = EA/L$, $F_1 = k(u_1 - u_2)$, $F_2 = k(u_2 - u_1)$, $F_3 = EA(u_3 - u_4)/L$, $F_4 = EA(u_4 - u_3)/L$



- The displacement constraints due to the left edge welding, displacement of the right end wall, and integrity of structure at the connection of the structural parts are $u_1 = 0$, $u_4 = a$, and $u_2 = u_3$.

- The force constraints are due to Newton **III** which requires that F_2 and F_3 are equal in magnitude and opposite in signs or $F_2 + F_3 = 0$.

- Altogether, the 8 equations determining the 4 displacement components u_1, u_2, u_3, u_4 and the 4 force components F_1, F_2, F_3, F_4 are given by

$$F_1 = k(u_1 - u_2),\ F_2 = k(u_2 - u_1),\ F_3 = k(u_3 - u_4),\ F_4 = k(u_4 - u_3),$$

$$u_1 = 0,\ \ u_4 = a,\ \ u_2 = u_3,\ \ F_2 + F_3 = 0.$$

- The linear equation system can be solved, e.g., by considering the equations in a proper order (to be discussed later in more detail), by Gauss elimination, by Mathematica, …

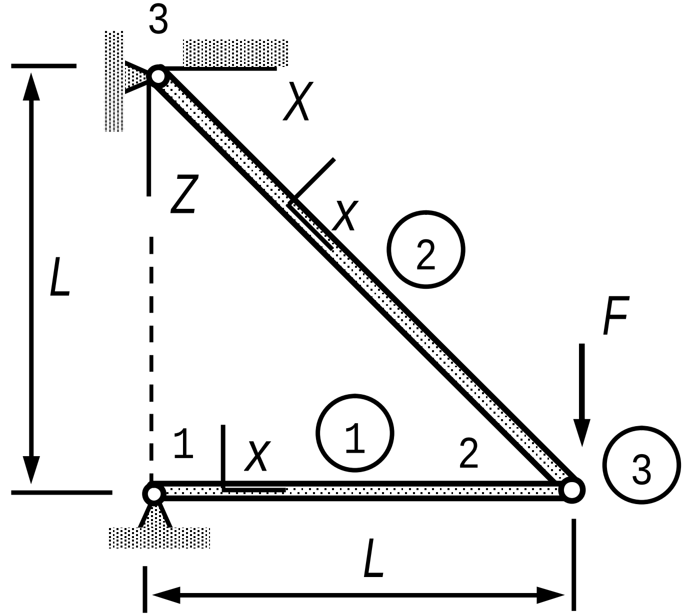
$$u_1 = 0,\ \ u_2 = \frac{1}{2}a,\ u_3 = \frac{1}{2}a,\ \ u_4 = a, \qquad \Leftarrow$$

$$F_1 = -\frac{1}{2}ka,\ \ F_2 = \frac{1}{2}ka,\ \ F_3 = -\frac{1}{2}ka,\ F_4 = \frac{1}{2}ka. \qquad \Leftarrow$$

1.4 FE-CODE OF MEC-E1050

	model	properties	geometry
1	BAR	{{E}, {A}}	Line[{1, 2}]
2	BAR	$\{\{\mathbb{E}\}, \{2\sqrt{2}\ A\}\}$	Line[{3, 2}]
3	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, L}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, L}	{uX[2], 0, uZ[2]}	{0, 0, 0}
3	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}



STRUCTURE

"Structure is a collection of *elements* (earlier structural parts) connected by *nodes* (earlier connection points). Displacement of the structure is defined by nodal translations and rotations of which some are known and some unknown."

$prb = \{ele, fun\}$ where

$ele = \{prt_1, prt_2, \ldots\}$ .... elements

$fun = \{val_1, val_2, \ldots\}$ .... nodes

Elements

$prt = \{typ, pro, geo\}$ where

$typ = \mathrm{BAR} \,|\, \mathrm{TORSION} \,|\, \mathrm{BEAM} \,|\, \mathrm{RIGID} | \ldots |$ .... model

$pro = \{p_1, p_2, \ldots, p_n\}$ ....properties

$geo = \text{Point}[\{n_1\}]\,|\,\text{Line}[\{n_1, n_2\}]\,|\,\text{Triangle}[\{n_1, n_2, n_3\}]\,|\ldots|$ .... geometry

Nodes

$val = \{crd, tra, rot\}$ where

$crd = \{X, Y, Z\}$ .... structural coordinates

$tra = \{u_X, u_Y, u_Z\}$ .... translation components

$rot = \{\theta_X, \theta_Y, \theta_Z\}$ ....rotation components

ELEMENTS

Elements represent the structural parts modelled as solids, plates, beams, or rigid bodies or their simplified versions, external point and boundary forces and moments.

Constraint

$\{\text{JOINT},\{\}|\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\}\},\text{Point}[\{n_1\}]\}$ ....displacement constraint

$\{\text{JOINT},\{\},\text{Line}[\{n_1,n_2\}]\}$ ....displacement constraint

$\{\text{RIGID},\{\}|\{\{\underline{u}_X,\underline{u}_Y,\underline{u}_Z\},\{\underline{\theta}_X,\underline{\theta}_Y,\underline{\theta}_Z\}\},\text{Point}[\{n_1\}]\}$ ... displacement/rotation constraint

$\{\text{RIGID},\{\},\text{Line}[\{n_1,n_2\}]\}$ .... rigid constraint

$\{\text{SLIDER},\{n_X,n_Y,n_Z\},\text{Point}[\{n_1\}]\}$ ....slider constraint

Force

$\{\text{FORCE},\{F_X,F_Y,F_Z\},\text{Point}[\{n_1\}]\}$ .... point force

$\{\text{FORCE},\{F_X,F_Y,F_Z,M_X,M_Y,M_Z\},\text{Point}[\{n_1\}]\}$ .... point load

$\{\text{FORCE},\{f_X, f_Y, f_Z\},\text{Line}[\{n_1,n_2\}]\}$ .... distributed force

$\{\text{FORCE},\{f_X, f_Y, f_Z\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ .... distributed force

Beam model

$\{\text{BAR},\{\{E\},\{A\},\{f_X, f_Y, f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....bar mode

$\{\text{TORSION},\{\{G\},\{A\},\{\{m_X,m_Y,m_Z\}\}\},\text{Line}[\{n_1,n_2\}]\}$............................ torsion mode

$\{\text{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz}\},\{f_X, f_Y, f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....beam

$\{\text{BEAM},\{\{E,G\},\{A,I_{yy},I_{zz},\{j_X, j_Y, j_Z\}\},\{f_X, f_Y, f_Z\}\},\text{Line}[\{n_1,n_2\}]\}$ ....beam

Plate model

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$.......................thin slab mode

$\{\text{PLANE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3,n_4\}]\}$..................thin slab mode

$\{\text{PLATE},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$ ....bending mode

$\{\text{SHELL},\{\{E,\nu\},\{t\},\{f_X, f_Y, f_Z\}\},\text{Polygon}[\{n_1,n_2,n_3\}]\}$.......................................plate

Solid model

$\{\text{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\},\text{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

$\{\text{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z\}\},\text{Hexahedron}[\{n_1,n_2,n_3,n_4,n_5,n_6,n_7,n_8\}]\}$ ....solid

$\{\text{SOLID},\{\{E,\nu\},\{f_X, f_Y, f_Z, m_X, m_Y, m_Z,\}\},\text{Tetrahedron}[\{n_1,n_2,n_3,n_4\}]\}$ ....solid

OPERATIONS

Operations act on structure as defined by prb. The main operations of MEC-E1050 are solving the unknowns in displacement analysis and displaying the problem definition in a formatted form.

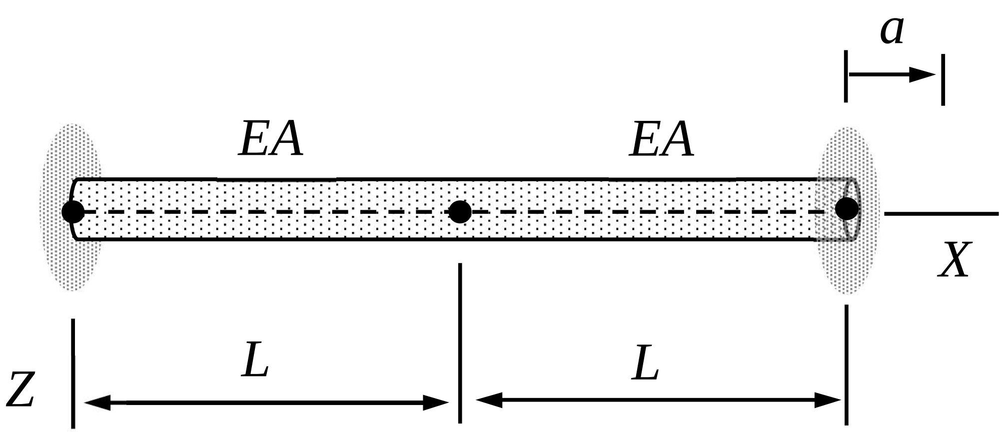
$prb = \text{REFINE}[prb]$ ....refine structure representation

$\text{Out} = \text{FORMATTED}[prb]$ .... display problem definition

$\text{Out} = \text{STANDARDFORM}[prb]$ .... display virtual work expression

$sol = \text{SOLVE}[prb]$ ....solve the unknowns

EXAMPLE 1.1. A connector bar is welded at its ends to rigid walls. If the right end wall displacement is a, determine the displacements of connection points 1, 2, and 3 and the forces acting on structural parts. Cross sectional area A and Young's modulus of the material E are constants and the displacement force relationship of a bar is the same as that of a spring with coefficient $k = EA / L$. Model the structure as a collection of two bars (1 and 2).



Answer $u_1 = 0,\ u_2 = u_3 = \frac{1}{2}a,\ u_4 = a,\ F_1 = F_3 = -\frac{1}{2}ka,\ F_2 = F_4 = \frac{1}{2}ka.$

- Problem description by $prb = \{ele,\ fun\}$ and two operations acting on it

```
ele = {
   {RIGID, {{0, 0, 0}, {0, 0, 0}}, Point[{1}]},
   {BAR, {{E}, {A}}, Line[{1, 2}]},
   {RIGID, {}, Line[{2, 3}]},
   {BAR, {{E}, {A}}, Line[{3, 4}]},
   {RIGID, {{a, 0, 0}, {0, 0, 0}}, Point[{4}]}};

fun = {
   {{0, 0, 0}, {uX[1], 0, 0}, {0, 0, 0}},
   {{L, 0, 0}, {uX[2], 0, 0}, {0, 0, 0}},
   {{L, 0, 0}, {uX[3], 0, 0}, {0, 0, 0}},
   {{2 L, 0, 0}, {uX[4], 0, 0}, {0, 0, 0}}};

FORMATTED[{ele, fun}]

SOLVE[{ele, fun}]
```

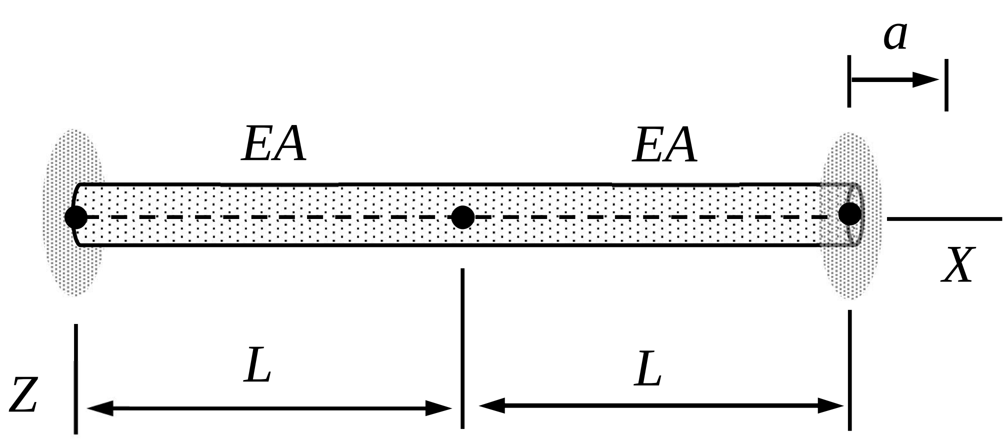
- Outcome of the operations is the structure description in table format and solution to the unknowns of the displacement problem (in format of a rule)

	model	properties	geometry
1	RIGID	{{0, 0, 0}, {0, 0, 0}}	Point[{1}]
2	BAR	{{E}, {A}}	Line[{1, 2}]
3	RIGID	{}	Line[{2, 3}]
4	BAR	{{E}, {A}}	Line[{3, 4}]
5	RIGID	{{a, 0, 0}, {0, 0, 0}}	Point[{4}]

	$\{X,Y,Z\}$	$\{u_X,u_Y,u_Z\}$	$\{\theta_X,\theta_Y,\theta_Z\}$
1	{0, 0, 0}	{uX[1], 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, 0, 0}
3	{L, 0, 0}	{uX[3], 0, 0}	{0, 0, 0}
4	{2 L, 0, 0}	{uX[4], 0, 0}	{0, 0, 0}

$$\left\{FX[1] \to -\frac{a\,A\,E}{2\,L},\ FX[4] \to \frac{a\,A\,E}{2\,L},\ FX[\{3,2\}] \to -\frac{a\,A\,E}{2\,L},\ uX[1] \to 0,\ uX[2] \to \frac{a}{2},\ uX[3] \to \frac{a}{2},\ uX[4] \to a\right\}$$

EXAMPLE 1.2. A connector bar is welded at its ends to rigid walls. If the right end wall displacement is a, determine the displacement of point 2. Cross sectional area A and Young's modulus of the material E are constants. Model the structure as a collection of two bars (1 and 2).



Answer $u_2 = \frac{1}{2}a$ (Mathematica notebook)

PREREQUISITE: MATRIX ALGEBRA I

Addition	$\mathbf{C} = \mathbf{A} + \mathbf{B}$	$C_{ij} = A_{ij} + B_{ij}$
Multiplication (scalar)	$\mathbf{C} = \alpha \mathbf{A}$	$C_{ij} = \alpha A_{ij}$
Multiplication (matrix)	$\mathbf{C} = \mathbf{AB}$	$C_{ij} = \sum_{k \in \{1 \ldots n\}} A_{ik} B_{kj}$

Unit matrix	$\mathbf{I}$	$\delta_{ij} = 1 \quad i = j, \quad \delta_{ij} = 0 \quad i \neq j$
Symmetric matrix	$\mathbf{A} = \mathbf{A}^{\mathrm{T}}$	$A_{ij} = A_{ji}$
Skew symmetric matrix	$\mathbf{A} = -\mathbf{A}^{\mathrm{T}}$	$A_{ij} = -A_{ji}$
Positive definite matrix	$\mathbf{x}^{\mathrm{T}} \mathbf{A} \mathbf{x} > 0 \quad \forall \mathbf{x} \neq 0$	

PREREQUISITE: MATRIX ALGEBRA II

Transpose	$\mathbf{A}^{\mathrm{T}}$	$A_{ij}^{\mathrm{T}} = A_{ji}$
Inverse	$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$	$\sum_{k \in \{1 \ldots n\}} A_{ik} A_{kj}^{-1} = \delta_{ij}$
Derivative	$\dot{\mathbf{x}}$	$\dot{x}_i = dx_i / dt$

Linear equation system	Find $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \mathbf{b}$
Eigenvalue problem	Find all $(\lambda, \mathbf{x})$ such that $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$
Eigenvalue composition	$\mathbf{A} = \mathbf{X} \boldsymbol{\lambda} \mathbf{X}^{-1}$, where $\mathbf{X} = [\mathbf{x}_1 \ldots \mathbf{x}_n]$ and $\boldsymbol{\lambda} = diag[\lambda_1 \quad \ldots \quad \lambda_n]$
Matrix function	If $\mathbf{A} = \mathbf{X} \boldsymbol{\lambda} \mathbf{X}^{-1}$, then $f(\mathbf{A}) = \mathbf{X} f(\boldsymbol{\lambda}) \mathbf{X}^{-1}$

EXAMPLE. Determine the square $\mathbf{A}^2$ and inverse $\mathbf{A}^{-1}$ of $\mathbf{A}$ if

$$\mathbf{A} = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix} \quad \left(\text{note: } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \quad \mathbf{A}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$$

Matrix squared $\quad \mathbf{A}^2 = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 23 & 16 \\ -8 & 7 \end{bmatrix}$ ←

Inverse matrix $\quad \mathbf{A}^{-1} = \begin{bmatrix} 5 & 2 \\ -1 & 3 \end{bmatrix}^{-1} = \frac{1}{17}\begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix}$ ←

From the viewpoint of computational complexity, solving a system of linear equations $\mathbf{Ax} = \mathbf{b}$ by Gauss elimination makes more sense than using the matrix inverse with $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$!

PREREQUISITE: MATRIX ALGEBRA III

Partitioned matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{bmatrix} \quad \& \quad \mathbf{B} = \begin{Bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{Bmatrix}$$

block or sub-matrix

Transpose

$$\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} \mathbf{a}_{11}^{\mathrm{T}} & \mathbf{a}_{21}^{\mathrm{T}} \\ \mathbf{a}_{12}^{\mathrm{T}} & \mathbf{a}_{22}^{\mathrm{T}} \end{bmatrix}$$

Multiplication

$$\mathbf{AB} = \begin{bmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{a}_{11}\mathbf{b}_1 + \mathbf{a}_{12}\mathbf{b}_2 \\ \mathbf{a}_{21}\mathbf{b}_1 + \mathbf{a}_{22}\mathbf{b}_2 \end{Bmatrix}$$

The rules are the same as with the ordinary matrices. The sizes of the blocks need to be consistent in operations like transposing and multiplication!

EXAMPLE. Determine the displacements w_i, if $i \in \{1,2,3\}$ the vector of displacements $\mathbf{a}$, stiffness matrix $\mathbf{K}$, and the loading vector $\mathbf{F}$ of the equilibrium equations $\mathbf{Ka} - \mathbf{F} = 0$ are given by

$$\mathbf{a} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}, \quad \mathbf{K} = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \text{ and } \mathbf{F} = P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

Answer $w_1 = \dfrac{P}{k}$, $w_2 = 2\dfrac{P}{k}$, and $w_3 = 3\dfrac{P}{k}$.

- With linear equation systems of more than two unknows, using a matrix inverse is not practical. Gauss elimination is based on row operations aiming at an upper diagonal matrix. After that, solution for the unknowns is obtained step-by-step starting from the last equation. In the present problem

$$k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

- Let us multiply the 2:nd equation by 2 and add to it equation 1 to get

$$k\begin{bmatrix} 2 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix} \quad \Rightarrow \quad k\begin{bmatrix} 2 & -1 & 0 \\ 0 & 3 & -2 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} = P\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix}.$$

- Let us multiply 3:rd equation by 3 and add to it the 2:nd equation to get the upper triangular matrix.

$$k\begin{bmatrix}2 & -1 & 0\\ 0 & 3 & -2\\ 0 & -3 & 3\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}=P\begin{Bmatrix}0\\ 0\\ 3\end{Bmatrix} \Rightarrow k\begin{bmatrix}2 & -1 & 0\\ 0 & 3 & -2\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}w_1\\ w_2\\ w_3\end{Bmatrix}=P\begin{Bmatrix}0\\ 0\\ 3\end{Bmatrix}.$$

- After these steps, solution is obtained step-by-step starting from the last equation:

$$kw_3 = 3P \quad \Leftrightarrow \quad w_3 = 3\frac{P}{k},$$ ←

$$k(3w_2 - 2w_3) = 0 \quad \Rightarrow \quad w_2 = \frac{2}{3}w_3 = 2\frac{P}{k},$$ ←

$$k(2w_1 - w_2) = 0 \quad \Rightarrow \quad w_1 = \frac{1}{2}w_2 = \frac{P}{k}.$$ ←