Aalto University
School of Science

Lecture 7: Collisions and Transport

Today's menu: weakly ionized gases

- Mean-free-path and collision frequency
- Mobility and *diffusion*
- Fick's law
- Sources & sinks: ionization & recombination
- Ambipolarity
- Decay times and steady-states
- Random walk and *diffusion*

Leaking out …

- In real world, every vessel leaks
- So far we have assumed perfect confinement and infinite plasma
- In reality, plasma is finite ➔ it has to have gradients
- Nature does not like gradients

  ➔ ***diffusion*** from high to low density

What drives diffusion?

Collisions

Collisions in weakly ionized plasma

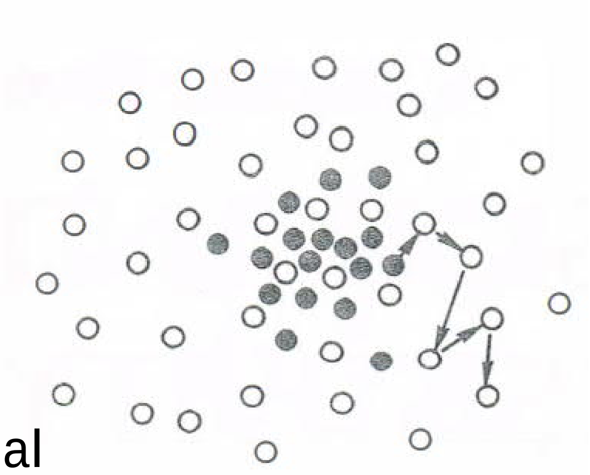
Weakly ionized plasmas – but why?

In fully ionized plasmas, collisions are *non-linear* effects

➔ Mathematically complicated

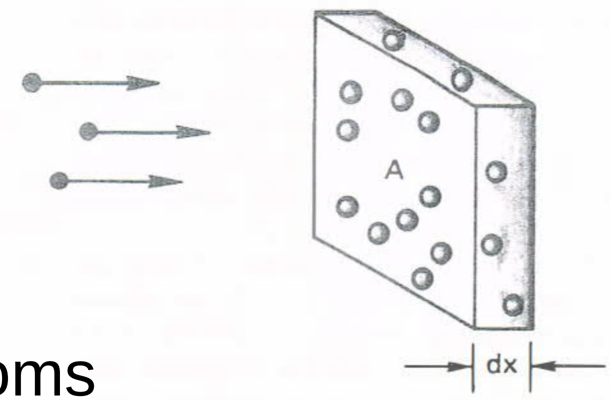
➔ Let's start with an easier case:

- ➔ Study collisions in *weakly* ionized plasma
- ➔ Charged particles suffer *head-on* collisions with neutral particles



Example: ionospheric plasma, $\frac{n_e}{n_n} \sim 10^{-6} - 10^{-3}$

Effect of collisions on flux



Flux Γ passes through a dense gas ➔ Γ'

- Dense gas consists of *scattering centers* = atoms
- Probability of colliding (= scattering of the flux) given by the *cross section* σ, which is the 'effective size' of an atom
- # of scatterers in a slab: $N = n_n \cdot A \cdot dx$
- Scatterers cover the fractional area $\frac{A_s}{A} = \frac{N \cdot \sigma}{A} = n_n \sigma dx$

➔ $\Gamma' = \Gamma - \Gamma \cdot \frac{N\sigma}{A} = \Gamma(1 - n_n \sigma dx)$

'Freedom' parameters for plasma particles

$$\Gamma' - \Gamma = -\Gamma n_n \sigma dx \rightarrow \frac{d\Gamma}{dx} = -n_n \sigma \Gamma$$
$$\Gamma(x) = \Gamma_0 e^{-n_n \sigma x} \equiv \Gamma_0 e^{-x/\lambda_{mfp}}$$

Here, $\lambda_{mfp} \equiv 1/n_n\sigma$ is called the *mean-free path* for collisions

A related quantity is the *mean time* between collisions: $\tau = \frac{\lambda_{mfp}}{v}$

But:

- plasma particles have a *distribution* of velocities
- Typically $\sigma = \sigma(\boldsymbol{v})$

➔ $< \sigma v > = \int v\sigma(\boldsymbol{v}) f(\boldsymbol{v}) d^3v$

➔ collision frequency: $\nu_{coll} = \frac{1}{\tau} = \frac{v}{\lambda_{mfp}} = n_n < \sigma v >$

Plasma motion due to collisions

Collisions cause friction ➔ have to be included in the EoM:

$$mn\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right] = qn\boldsymbol{E} - \nabla p - mn\nu_{coll}\boldsymbol{v}$$

Want to study effect of collisions *only* ➔ simplify other stuff away:

1. Steady state
2. Low flow = assume friction dominates
3. Isothermal, $T = const$

➔ $\boldsymbol{v} = (qn\boldsymbol{E} - T\nabla n)\,/mn\nu_{coll} = \frac{q}{m\nu_{coll}}\boldsymbol{E} - \frac{T}{m\nu_{coll}}\frac{\nabla n}{n}$

Diffusion in weakly ionized plasma

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Our first transport coefficients …

➔ In the presence of collisions with *neutrals*, our plasma fluid moves according to the *density gradient* and *electric field*:

$$\Gamma_j = n\boldsymbol{v}_j = \pm\mu_j n\boldsymbol{E} - D_j \nabla n$$

where

$$\mu_j \equiv \frac{q_j}{m_j \nu_{coll}}$$ is called the *mobility* of the plasma

$$D_j \equiv \frac{T_j}{m_j \nu_{coll}}$$ is the *diffusion coefficient*

Important observations:

1. The flux is thus driven by *gradients*, as initially assumed:

$$\boldsymbol{\Gamma}_j = \mp \mu_j n \nabla \phi - D_j \nabla n$$

2. Collisions result into *diffusion* and diffusion in the presence of collisions means *transport*

Fick's law

For diffusion in regular gases the *Fick's law* applies

$$\boldsymbol{\Gamma} = -D\nabla n$$

The physics of Fick's law:

- *Nature likes to flatten out gradients*

or, to put it in another way,

- *Gradients* drive fluxes.

A weakly ionized plasma thus obeys Fick's law ($\boldsymbol{E} = 0$):

$$\Gamma_j = -D_j \nabla n$$

What is the time scale of flattening?

Fluids obey continuity equation:

$$\frac{\partial n_j}{\partial t} + \nabla \cdot \boldsymbol{\Gamma}_\mathrm{j} = 0$$

Quasineutrality ➔ $n_i \approx n_e \approx n$ ➔ $\nabla \cdot \boldsymbol{\Gamma}_e \approx \nabla \cdot \boldsymbol{\Gamma}_i$

How about the individual fluxes?

Assume $\boldsymbol{\Gamma}_e \neq \boldsymbol{\Gamma}_i$

➔ charge imbalance

➔ electric field sufficient to retard electrons & accelerate ions to make $\boldsymbol{\Gamma}_e = \boldsymbol{\Gamma}_i$.

Ambipolar stuff …

Find the magnitude of this *ambipolar electric field*:

$$\boldsymbol{\Gamma}_e = \boldsymbol{\Gamma}_i \rightarrow \mu_i n\boldsymbol{E} - D_i \nabla n = -\mu_e n\boldsymbol{E} - D_e \nabla n$$

➔ $\boldsymbol{E} = \frac{D_i - D_e}{\mu_i + \mu_e} \frac{\nabla n}{n}$

➔ The flux of the *plasma* is given by

$\Gamma = \Gamma_i = \mu_i \frac{D_i - D_e}{\mu_i + \mu_e} \nabla n - D_i \nabla n = -\frac{\mu_e D_i + \mu_i D_e}{\mu_i + \mu_e} \nabla n$; *Fick's law again!*

We have *ambipolar* fluxes driven by *ambipolar diffusion coefficient*

$$D_a \equiv \frac{\mu_e D_i + \mu_i D_e}{\mu_i + \mu_e} \approx D_i + \frac{\mu_i}{\mu_e} D_e \approx D_i + \frac{T_e}{T_i} D_i = D_i \left(1 + \frac{T_e}{T_i}\right)$$

Usually $\mu_e \gg \mu_i$

Decay time of *weakly ionized* plasma

Now we have continuity equation for *the plasma*:

$$\frac{\partial n}{\partial t} + \nabla \cdot \boldsymbol{\Gamma} = \frac{\partial n}{\partial t} - D_a \nabla^2 n = 0$$

Recall Schrödinger eqn ➔ *separation of variables*: $n(\boldsymbol{r}, t) = X(\boldsymbol{r})T(t)$

Let's try to solve this in two simple geometries:

1. 1D case, i.e., *slab geometry*
2. 2D case, i.e., *cylindrical geometry*

Plasma decay time in slab geometry

Substitute trial fct to 1D continuity equation: $X(x)\frac{dT}{dt} - D_a T\frac{d^2X}{dx^2} = 0$

➔ $\frac{1}{T}\frac{dT}{dt} = \frac{D_a}{X}\frac{d^2X}{dx^2} = const \equiv -\frac{1}{\tau}$

➔ $T(t) = n_0 e^{-t/\tau}$

➔ $\frac{d^2X}{dx^2} + \frac{1}{D_a\tau}X = 0 \rightarrow X(x) = A\sin kx + B\cos kx$, where $k^2 \equiv \frac{1}{D_a\tau}$

Plasma is bounded. Let boundaries be at $x = \pm L \rightarrow k = l\pi/2L$

➔ $n(x,t) = n_0 e^{-t/\tau}\cos\frac{\pi x}{2L}$, why only $l = 1$???

➔ the decay time is given by the diffusion coefficient: $\tau = \left(\frac{2L}{\pi}\right)^2 \frac{1}{D_a}$

Sanity checks …

Observations on τ :

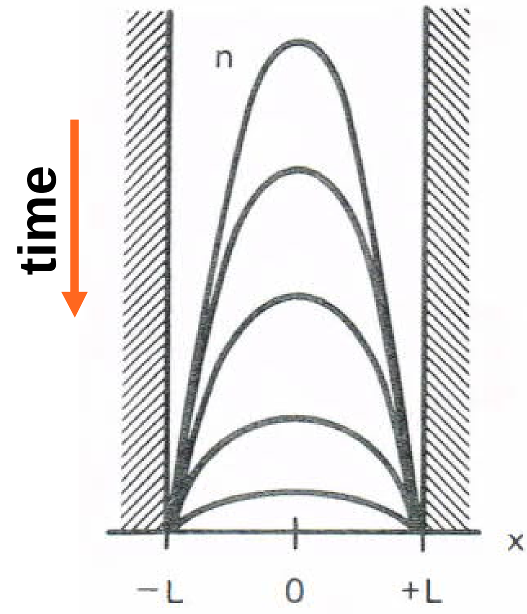
- τ increases with the box size L
- τ decreases with increasing diffusion

Makes sense. ☺

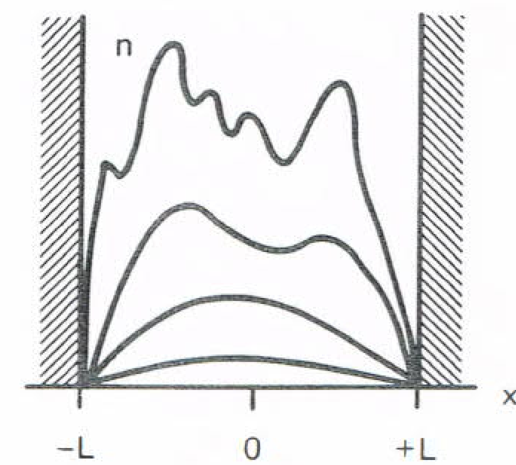
Also the shape of the solution, the *lowest diffusion mode*, looks reasonable, peaking at the center.

➔

Weakly ionized plasma decays exponentially at rate determined by its size and the diffusion coefficient



The decay process



Start with an abitrary initial shape

FT ➔ $n(x,0) = n_0\left[a_0 + \sum a_l \cos\frac{\left(l+\frac{1}{2}\right)\pi x}{L} + \sum b_m \sin\frac{m\pi x}{L}\right]$

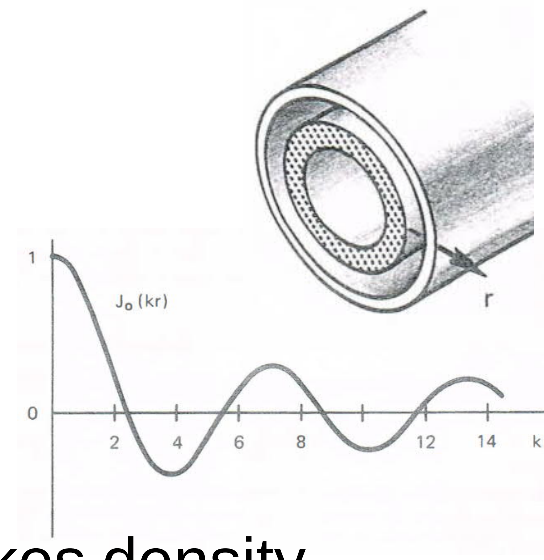
➔ Trial solution:

$$n(x,t) = n_0\left[a_0 e^{-t/\tau_0} + \sum a_l \cos\frac{\left(l+\frac{1}{2}\right)\pi x}{L} e^{-t/\tau_l} + \sum b_m \sin\frac{m\pi x}{L} e^{-t/\tau_m}\right]$$

Substitute to the diffusion equation ➔ $1/\tau_l = D_a\left[\left(l+\frac{1}{2}\right)\pi/L\right]^2$

➔ $\tau_l = \left[(l+\frac{1}{2})\pi/L\right]^{-2} 1/D_a$ ➔ finest structures decay fastest!

Getting more realistic: Decay of a cylindrical plasma



Assume cylindrical symmetry ➔ $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$

Separate variables ➔ $\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} + \frac{1}{D\tau}X = 0$

In cylindrical geometry, the volume increase in r makes density drop faster ➔ could expect something like decaying cosine

Indeed, solutions are *Bessel functions!* Here, $J_0(r)$ *!*

B.C's at $r = 0, r = a$ ➔ $\frac{a}{\sqrt{D_a\tau}} = 2.4$ (first zero of J_0) ➔ $\tau = \left(\frac{a}{2.4}\right)^2 \frac{1}{D_a}$

How to get steady-state plasma…

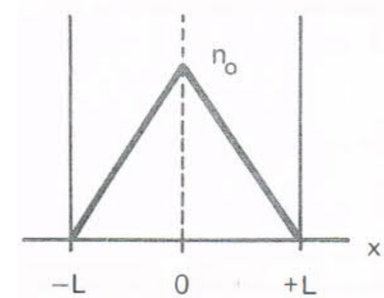
… if plasma unavoidably decays due to inter-particle interactions?

Need a particle source, $S_+(\boldsymbol{r})$!!

Ways to 'feed' a plasma:

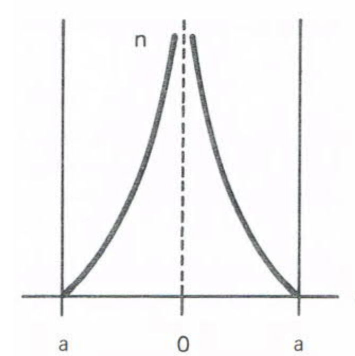
- Injection of particles
- Puffing of particles
- (recycling of particles – more about this later)

Simple steady-state cases: 1. local sources



1-D case: a plane source at $x = 0$: $S_+(x) = S_+\delta(0)$

➔For $x \neq 0$: $\frac{\partial^2 n}{\partial x^2} = 0 \rightarrow n(x) = n_0(1 - \frac{|x|}{L})$



2-D case: cylindrical plasma, line source at $r = 0$.

- (e.g., beam of energetic electrons causing ionization along the axis)

For $r \neq 0$: $\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial n}{\partial r} = 0 \rightarrow n(r) = n_0 \log \frac{a}{r}$, where $n(a) = 0$ was used

Simple steady-state cases: 2. ionization source

Plasma can be fuelled also by a *heat* source (in cold plasmas): electrons in the hot Maxwellian tail keep ionizing the gas neutrals

a 'continuous' source (around heat source): $S_+ \propto n$.

Let's write then $S_+(\boldsymbol{r}) = Zn(\boldsymbol{r})$, where $Z \neq Z(\boldsymbol{r})$ is the *ionization* fct

➔ $\nabla^2 n = -\frac{Z}{D} n$

But this is formally the same as the eqn for $X(r)$ ➔ $n(r) = J_0(r)$

How about *sinks?*

We just had *ionization* as a source.

The reverse process, *recombination*, is a sink, S_- .

Recombination requires both electrons and ions ➔ $S_- \propto n_i n_e$.

Study the effect of recombination alone = neglect diffusion

➔ $\frac{\partial n}{\partial t} = -\alpha n^2$, where α is the recombination coefficient, $\alpha \neq \alpha(n)$

Non-linear equation! ➔ separation of variables not possible

➔ solution by ’eye-balling’: $\frac{1}{n(r,t)} = \frac{1}{n_0(r)} + \alpha t$ *(HW: just show)*

New processes can change the character of the solutions

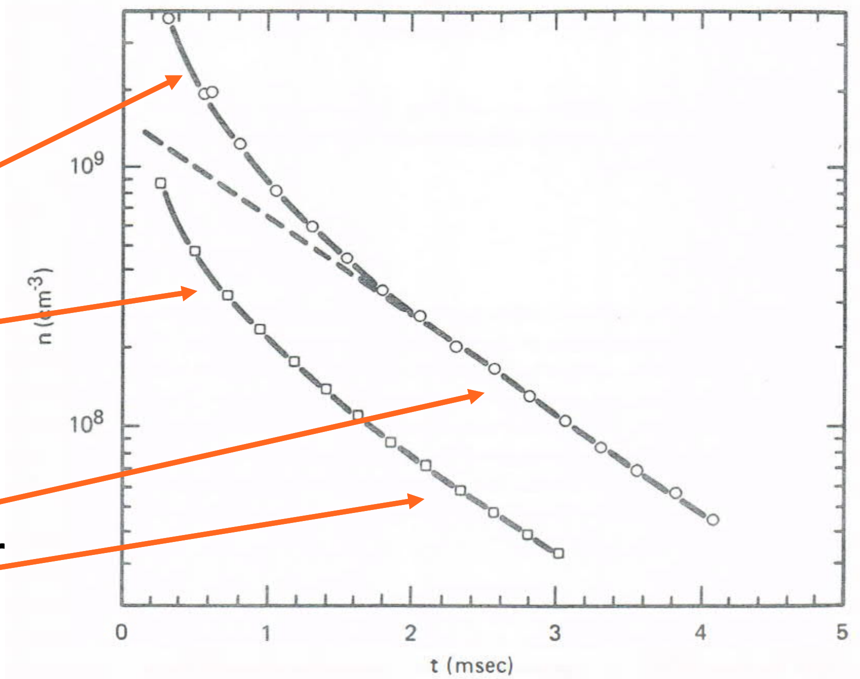
At high density, recombination ($\propto n^2$) typically dominates

➔ $n(r,t) \propto \frac{1}{\alpha t}$

and the density falls *reciprocally* in time, *not* exponentially!

As the density drops, diffusion takes over

➔ exponential decay



Until now, we have been studying ’freely floating’ plasmas

But mostly we are interested in *magnetized* plasmas!

How does the plasma decay when it is imbedded in a confining magnetic field?

Like *fusion* or *atmospheric* or *solar* plasmas…

What does the magnetic field do in weakly ionized plasmas?

In direction parallel to $\boldsymbol{B}$, magnetic field has no say

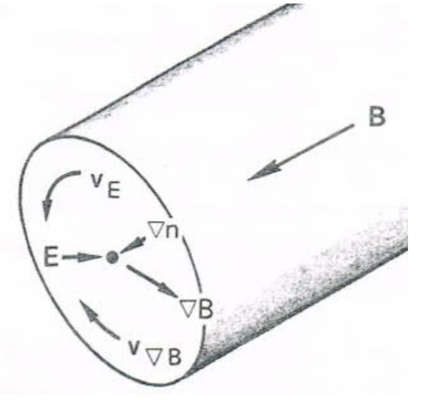
➔ same physics as before

What is interesting is the transport *perpendicular* to $\boldsymbol{B}$.

These particles are glued to the fieldlines.

… But we can have cross-field drifts! $\boldsymbol{ExB}$ & *Co*!

Luckily drifts can be aligned so that they are parallel to walls (laboratory plasmas)



Analyze fluid equations $\perp \boldsymbol{B} = B_0\hat{\boldsymbol{z}}$

Same simplifying assumptions as before ➔

Motion $\perp \boldsymbol{B}$: $mn\frac{d\boldsymbol{v}_\perp}{dt} \approx 0 \approx nq(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) - T\nabla n - mn\nu_{coll}\boldsymbol{v}_\perp$

$$v_x = \pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} \pm \frac{\Omega}{\nu_{coll}} v_y$$

$$v_y = \mp\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} \mp \frac{\Omega}{\nu_{coll}} v_x$$

➔ HW: $$\boldsymbol{v}_\perp = \pm\mu_\perp \boldsymbol{E} - D_\perp \frac{\nabla_\perp n}{n} + \frac{\boldsymbol{v}_{E\times B} + \boldsymbol{v}_{\text{dia}}}{1 + \nu_{coll}^2/\Omega^2},$$

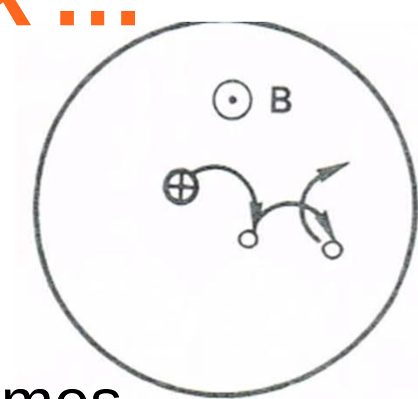
where $\mu_\perp \equiv \mu/(1 + \Omega^2\tau_{coll}^2)$ and $D_\perp \equiv D/(1 + \Omega^2\tau_{coll}^2)$

Physics of $v_\perp$

1. Familiar magnetic drifts perpendicular to their respective gradients ($\boldsymbol{v}_{E\times B} \propto \nabla\phi, \boldsymbol{v}_{dia} \propto \nabla n$), but slowed down by collisions with neutrals by the *drag factor* $1 + \nu_{coll}^2/\Omega^2$.
   - Increase magnetic field and/or reduce neutral density ➔ good old drifts!

2. Mobility drift parallel to $\boldsymbol{E}$ and diffusion drift parallel to ∇n, obtained in the absence of $\boldsymbol{B}$ are now slowed down by the factor $1 + \Omega^2\tau_{coll}^2$
   - This is *not* the same as the drag factor but works the opposite way (as it should): increase magnetic field and/or reduce neutral density ➔ mobility and diffusion drifts vanish

More on physics of $v_\perp$ -- random walk …



$\Omega\tau_{coll} \ll 1$ ➔ B-field has little effect on diffusion

$\Omega\tau_{coll} \gg 1$ ➔ B-field reduces diffusion across $\boldsymbol{B}$

The physics of 'magnetic' slowing down of diffusion:

In the presence of strong $\boldsymbol{B}$ the diffusion coefficient becomes

$$D_\perp \rightarrow \frac{T}{m\nu_{coll}} \frac{1}{\Omega^2 \tau_{coll}^2} = \frac{T}{m\Omega^2} \nu_{coll}$$

We then realize: $\frac{T}{m\Omega^2} \sim \frac{v_{th}^2}{\Omega^2} = r_L^2 \rightarrow D_\perp \sim r_L^2\, \nu_{coll} \sim stepsize^2/colltime$

➔ the effect of $\boldsymbol{B}$ is to reduce the step size from mean-free-path to Larmor radius!

Differences to ’free-floating’ plasma

No $\boldsymbol{B}$-field (or parallel to it): collisions *retard* the motion

➔ $D \propto 1/\nu_{coll}$

Across the $\boldsymbol{B}$-field; collisions are *needed* for particles to jump from one Larmor orbit to another

➔ $D \propto \nu_{coll}$

Also the role of particle mass is reversed:

- No $\boldsymbol{B}$ (or ∥ $\boldsymbol{B}$): $D \propto 1/\sqrt{m}$; *light electrons move faster along* $\boldsymbol{B}$
- ⊥ $\boldsymbol{B}$: $D \propto \sqrt{m}$; *ions have larger Larmor radius = step size*