Aalto University
School of Science

Lecture 8:
Fully ionized plasma: diffusion and resistivity

Today's menu

- Coulomb scattering vs billiard balls
- 90-degree collision frequency
- Plasma resistivity
- Classical diffusion
- *Anomalous* diffusion and Bohm diffusion coefficient
- Freezing magnetic field
- Reconnection
- Magnetic field decay

How general are our results?

So far we have dealt with only weakly ionized plasmas.

But for most of the time we are interested in hotter plasmas (fusion, sun, …) when the plasma can be fully ionized …

Do our assumptions and results then hold?

Collisions in fully-ionized plasma

From head-on collisions to scattering…

In partially-ionized gases, the charged particles suffer predominantly head-on collisions with the gas neutrals:

- Electrons from the 'solid' electron cloud around the atom
- Ions whizz through the electron cloud and collide with the nucleus

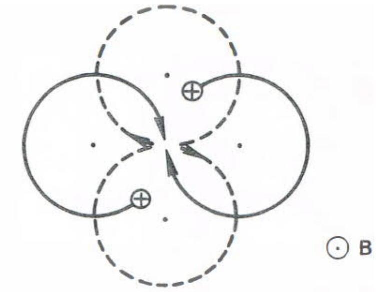
In fully-ionized plasma the situation is totally different:

The nature of interactions changes completely:

- continuous (weak) Coulomb *scattering* of charges from each other
- Plasma particle can scatter from one of its kind or from the other species ...

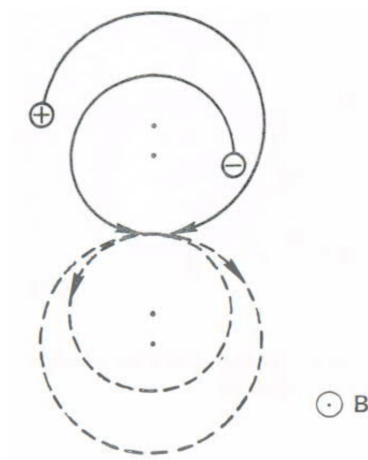
Physical picture of scattering in plasma

- Like-particle collisions: *no net motion since momentum & energy have to be conserved*
- Collisions from different species: *net motion allowed*



Physics of collisions different for electrons and ions:

- Electrons bounce off ions ➔ random walk
- Ions are only slightly disturbed in individual collision, net motion due to continuous bombardment of electrons.



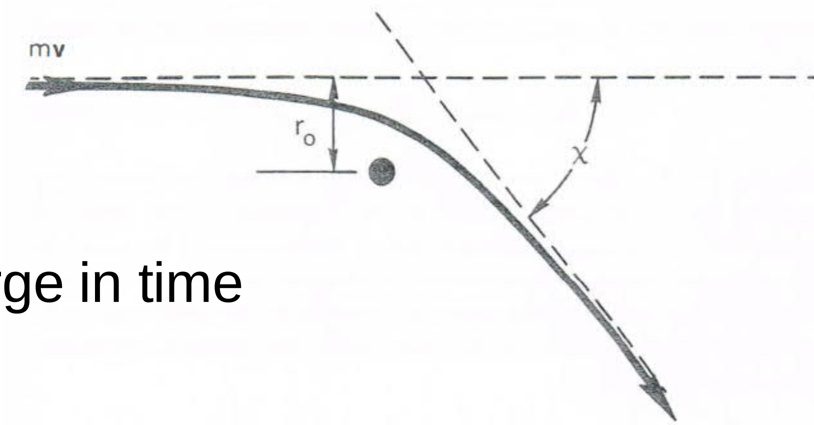
Fundamentally different from weakly ionized gas, where collision with a neutral *always* leads to diffusion

What happens in Coulomb collisions?

Coulomb collision is essentially *Rutherford scattering.*

Let's carry out a simplified analysis:

$$F_{coulomb} = -\frac{e^2}{4\pi\epsilon_0 r^2}$$



A particle w/ velocity v scatters off the other charge in time $\Delta t \sim r_0 / v$ ➔ $\Delta p \sim F\Delta t \sim e^2/4\pi\epsilon_0 r_0 v$

Assume large-angle collision ➔ $\Delta p \sim p = mv$

➔ $mv \sim e^2/4\pi\epsilon_0 r_0 v \leftrightarrow r_0 \sim e^2/4\pi\epsilon_0 m v^2$

Collision frequency in fully ionized plasma

So we can estimate the effective cross section:

$$\sigma \sim \pi r_0^2 \sim e^4/16\pi\epsilon_0^2 m^2 v^4$$; almost the Rutherford scattering formula!

➔ the frequency for *electrons colliding with ions*:

$$\nu_{ei} \sim n\sigma v \sim ne^4/16\pi\epsilon_0^2 m^2 v^3$$

Assume Maxwellian plasma: $mv^2 \sim T \rightarrow \nu_{ei} \sim ne^4/16\pi\epsilon_0^2\sqrt{m}T^{3/2}$.

But we have cheated: *large-angle Coulomb collisions are rare*.

Small-angle collisions dominate ➔ frequency for a 90° change in direction as a cumulative effect:

$$\nu_{ei} = \pi ne^4 \log\Lambda/(4\pi\epsilon_0)^2\sqrt{m}T^{3/2}$$

Featuring today: *electron collision frequency!*

Observations on the electron collision frequency:

1. $\nu_{ei} \propto n$. Intuitive and understandable
2. $\nu_{ei} \propto v^{-3}$. Interesting implications …

If the electron temperature is sufficiently high and an electric field is introduced to the plasma, for some high-energy electrons in the Maxwellian tail collisional friction becomes negligible

➔ these run-away electrons are accelerated to relativistic energies and form a *run-away electron beam* detached from the bulk plasma

Collisions & transport in fully-ionized plasma

Eqs of motion keeping only collisional effects *between* species:

$$Mn\frac{d\boldsymbol{v}_i}{dt} = en(\boldsymbol{E} + \boldsymbol{v}_i \times \boldsymbol{B}) - \nabla p_i + \boldsymbol{P}_{ie}$$
$$mn\frac{d\boldsymbol{v}_e}{dt} = -en(\boldsymbol{E} + \boldsymbol{v}_e \times \boldsymbol{B}) - \nabla p_e + \boldsymbol{P}_{ei}$$

$\boldsymbol{P}_{ei}$ and $\boldsymbol{P}_{ie}$ stand for the collisional exchange of momentum between electron and ion fluid ➔ $\boldsymbol{P}_{ei} = -\boldsymbol{P}_{ie}$

$\boldsymbol{P}_{ei}$ represents friction ➔ $\boldsymbol{P}_{ei} = mn(\boldsymbol{v}_i - \boldsymbol{v}_e)\nu_{ei}$

Collisions and resistivity …

On the physics basis, $\boldsymbol{P}_{ei}$ should reflect the Coulomb interaction (charges):

➔ $\boldsymbol{P}_{ei} = \eta e^2 n^2 (\boldsymbol{v}_i - \boldsymbol{v}_e)$, where η is an unknown factor.

Equating the two expressions we find $\eta = \frac{m}{ne^2} \nu_{coll}$

➔ $\eta = \frac{\pi \sqrt{m} e^2}{(4\pi\epsilon_0)^2 T^{3/2}} \log \Lambda$; *specific resistivity of plasma*

Why the name?

Look at simple, unmagnetized plasma: $en\boldsymbol{E} = \boldsymbol{P}_{ei} = \eta e^2 n^2 (\boldsymbol{v}_i - \boldsymbol{v}_e)$

$\boldsymbol{j} = en(\boldsymbol{v}_i - \boldsymbol{v}_e) \rightarrow \boldsymbol{E} = \eta \boldsymbol{j}$ and we recovered *Ohm's law !!!*

Observations on resistivity

1. Surprise: $\eta \neq \eta(n)$!

That is, it does not depend on the # of charge carriers!

On 2nd thought understandable:

- $\boldsymbol{j}$ increases with n_e
- $friction \propto n_i$

$n_e = n_i$ ➔ density dependences cancel out.

2. Resistivity decreases with T_e ➔

- So-called *Ohmic heating* is limited to about 1 keV
- Hot plasmas are very good conductors!

What if there is a magnetic field…?

With the presence of a magnetic field, not all directions are equal.

Since current is motion of electrons and motion across the field is hindered, one would expect $\eta_{\perp} > \eta_{\parallel}$.

This is indeed the case:

$\eta_{\perp} = 2\eta_{\parallel}$ & a factor of the type $\Omega^2\tau_{coll}^2$ to be included

Diffusion in fully-ionized plasma

Take the relevant MHD equations for the *whole* plasma in steady state:

$$0 = \boldsymbol{j} \times \boldsymbol{B} - \nabla p$$

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta \boldsymbol{j}$$

Where we have now included the resistivity.

Parallel direction:

$\nabla p = 0$ and $\boldsymbol{E} = \eta_{\parallel} \boldsymbol{j}$. Not very interesting

Perpendicular direction:

multiply by $\times \boldsymbol{B} \rightarrow \boldsymbol{E} \times \boldsymbol{B} + (\boldsymbol{v}_{\perp} \times \boldsymbol{B}) \times \boldsymbol{B} = \eta_{\perp} \boldsymbol{j} \times \boldsymbol{B} = \eta_{\perp} \nabla p$

➔ $\boldsymbol{E} \times \boldsymbol{B} - B^2 \boldsymbol{v}_{\perp} = \eta_{\perp} \nabla p$

$$➔ \ \boldsymbol{v}_{\perp} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} - \frac{\eta_{\perp}}{B^2} \nabla p$$

’Classical’ diffusion

The cross-field drift of fully-ionized plasma thus has 2 terms:

#1. The whole plasma drifts with the $\boldsymbol{E} \times \boldsymbol{B}$ drift velocity

#2. Cross-field drift whose origin is in the collisions (via η)

Let’s call the collisional drift $\boldsymbol{v}_D = -\frac{\eta_\perp}{B^2} \nabla p = -\frac{\eta_\perp T}{B^2} \nabla n$; (isothermal)

Then cross-field flux is: $\boldsymbol{\Gamma}_\perp = n\boldsymbol{v}_D = -\frac{\eta_\perp n T}{B^2} \nabla n$; we have Fick’s law!

➔ so-called *classical diffusion coefficient* $D_{cl} \equiv \frac{\eta_\perp n T}{B^2}$

Physics of classical diffusion coefficient

1. $D_{cl} \propto 1/B^2$, same as for weakly-ionized plasma
2. $D_{cl} \propto n$; this is different but understandable, makes the problem *non-linear …* ☹
3. $D_{cl} \propto T^{-1/2}$ since $\eta \propto T^{-3/2}$; so for fully-ionized plasmas, diffusion *reduces* with temperature. This is opposite to weakly-ionized case and has its origin in Coulomb cross section.
4. Diffusion is automatically ambipolar. This is due to $\boldsymbol{P}_{ei} = -\boldsymbol{P}_{ie}$
5. There is no mobility μ in fully ionized plasma. If a $\boldsymbol{E}_{\perp}$ appears ➔ both species just drift with the common $\boldsymbol{E} \times \boldsymbol{B}$ drift

Decay of fully ionized plasma

Now the tricky thing is that $D_{cl} \propto n$

➔ let's take n out: $D_{cl} = An$, where $A \equiv \eta T / B^2$

Equation of continuity: $\frac{\partial n}{\partial t} = \nabla \cdot (D_{cl} \nabla n) = A\nabla \cdot (2n\nabla n)$; $T_i + T_e = 2T$

➔ $\frac{\partial n}{\partial t} = A\nabla^2 n^2$; this is non-linear and no longer a diffusion equation

Take $n = T(t)S(\boldsymbol{r}) \rightarrow \frac{1}{T^2}\frac{dT}{dt} = \frac{A}{S}\nabla^2 S^2 = -1/\tau$

Time-dependence: $\frac{1}{T} = \frac{1}{T_0} + \frac{t}{\tau}$. Same decay pattern as w /recombination

➔ Fully ionized plasma decays *reciprocally* .

Simple steady-state profiles

Diffusion & recombination deplete plasma ➔ need a source.

Take simple line source on the axis that compensates losses.

Steady state ➔ for $r > 0$, diffusion and recombination determine profile

$$-A\nabla^2 n^2 = -\alpha n^2$$; linear equation for n^2 !!

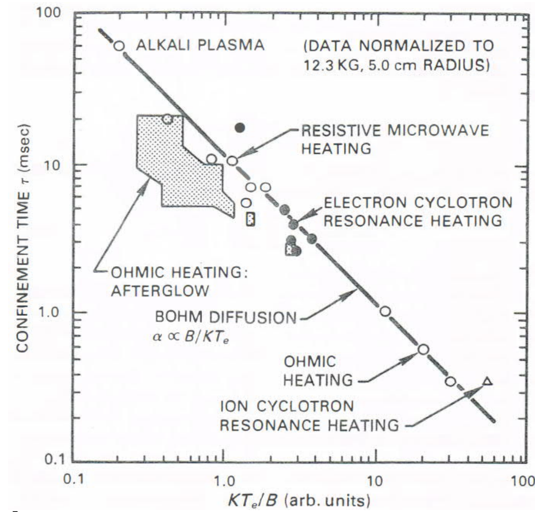
We have already solved such equations ➔

- Cylindrical plasma ➔ Bessel functions
- Slab geometry ➔ $n^2 = n_0^2 e^{-\sqrt{\alpha/A}\,r}$; easier to 'visualize'

➔ Fall-off length (scale length) of density: $L_{\mathrm{n}} = \sqrt{A/\alpha}$

Recall $A \propto 1/B^2$ ➔ strong magnetic field provides 'tighter package'

Let’s get real



It is time to compare our theoretical results to experiments.

First: compare *dependences* rather than absolute values.

Start with B-dependence (easy to control): $D_{cl} \propto 1/B^2$

… and we encounter an ugly experimental fact: $D_{cl} \propto 1/B$ (1)

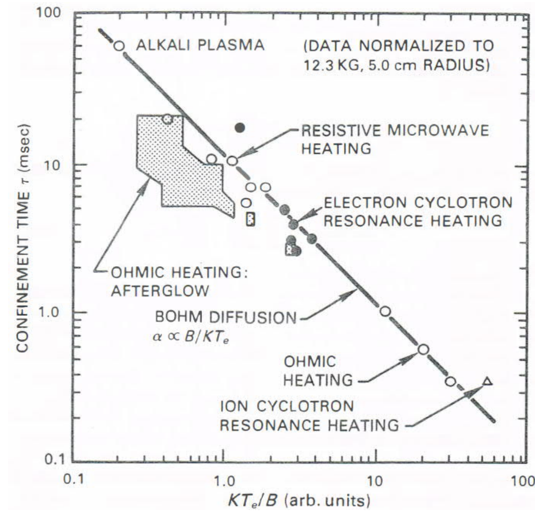
Not only that:

Plasma is found to decay *exponentially*, not reciprocally (2)

And comparing the numbers: $D_\perp \gg D_{cl}$ (3)   … like a factor of 10^4

(1) – (3) ➔ look for a *scaling law* based on experimental observations

Bohm diffusion …



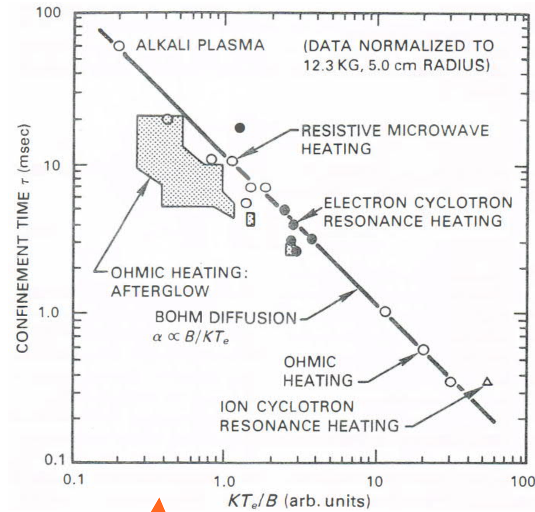
Semi-empirical formula for diffusion coefficient:

$$D_\perp = \frac{1}{16}\frac{T_e}{eB} \equiv D_B \text{ ; } \mathit{B \text{ for Bohm}}$$

No dependence of density

➔ exponential decay – as observed!

What is that plot there …?



Let's do a simple estimate for the decay time of the plasma

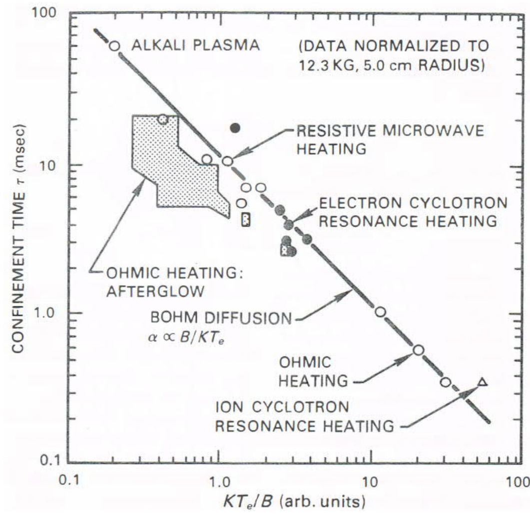
$$\frac{dN}{dt} \sim \frac{N}{\tau}$$

Cylinder ➔ $N \sim n\pi R^2 L$, $\frac{dN}{dt} \sim \Gamma_R 2\pi RL$

➔ $\tau \sim \frac{N}{dN/dt} \sim \frac{NR}{2\Gamma_R}$

Fick's law ➔ $\Gamma_R = D_B \frac{\partial N}{\partial R} \sim D_B \frac{N}{R}$ ➔ $\tau \equiv \tau_B = \frac{R^2}{2D_B} \propto \frac{1}{T/B}$

Why, oh why Bohm ???



But we had derived our own diffusion coefficient correctly!

Even with the simplifying approximations we should have captured the *essential* physics!

What can be wrong???

Many things can go wrong: potential candidates for D_B

1. Field errors:
   - The magnetic field is produced by a finite number of finite-size coils. Mis-alignment of the coils etc can lead to situations where field lines arrive at the vessel wall prematurely. Since λ_{mfp} is very long, even slight asymmetries in magnetic coils allow electrons to arrive to the wall – and pull ions along due to the ambipolar electric field
2. Unstable plasma waves:
   - The fluctuating EM fields can lead to *collisionless* random walk
3. Drift waves, called *convective cells*, exist ➔ *convective* transport

Hand-waving arguments for Bohm diffusion

Assume this *anomalous* (since it is not explained by theory) transport to be caused by $E \times B$ drifts caused by ”*something*”*.*

The cross-field flux then becomes

$$\Gamma = nv_{\perp} \propto n\frac{E}{B}$$

Let’s try to estimate the magnitude of the electric field:

Debye shielding ➔ $e\phi_{max} \sim T_e \rightarrow E = -\nabla\phi$

$$E_{max} \sim \phi_{max}/a \sim T_e/ea \quad ; \; a \sim \text{typical scale length}$$

➔ $\Gamma \sim \gamma \frac{n}{a}\frac{T_e}{eB} \sim \gamma \frac{T_e}{eB}\nabla n \propto -D_B \nabla n$ … but no hand-waving gives $\gamma = \frac{1}{16}$ …

About plasma resistivity

Jobs for resistivity

Somewhere along our collisional massaging the concept of *resistivity* popped up. What is it and what does it do?

Resistivity… resistance … current resisted … cables heating up … Joule heating ➔ *Ohmic heating* of plasma by electrical currents *!*

High school (in Finnish): 'Pirjo UI' = P = UI = RI^2 .

More fundamentally: $\frac{dw_B}{dt} = \eta j^2$, where w_B is the energy *density*

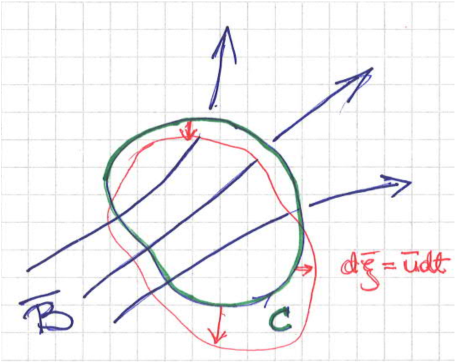
But this is not all that resistivity can do …

Resistivity can affect the magnetic topology of a plasma

Another look at the intimate relationship between magnetic field and plasma …

We know that individual charged particles are glued to field lines.

How about in the fluid picture: is there an unbreakable bond between plasma as a fluid and the magnetic field?



Consider magnetic flux through a surface S:

$$\Phi_B \equiv \int \boldsymbol{B} \cdot d\boldsymbol{S}$$

If the contour C is changing its shape with velocity $\boldsymbol{u}$, in time dt the area changes as $\boldsymbol{u}dt \times d\boldsymbol{l}$

➔ $$\frac{d\Phi_B}{dt} = \int \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{S} + \int \boldsymbol{B} \cdot (\boldsymbol{u} \times d\boldsymbol{l})$$

Frozen … *(NOT the Disney movie!)*

Use:

- Faraday's law
- the Stokes' theorem (with $\cdot \leftrightarrow \times$)

$$\rightarrow \frac{d\Phi_B}{dt} = \int \nabla \times (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) \cdot d\boldsymbol{S}$$

If $\boldsymbol{u} = \boldsymbol{v}$, i.e., the surface moves/expands/shrinks with the plasma: $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = 0$

➔ Within the framework of *ideal* MHD, the magnetic field is *frozen* to plasma:

$$\frac{d\Phi_B}{dt} = 0$$

i.e., the field lines have to follow the motion of the plasma.

Resistive MHD and Reconnection

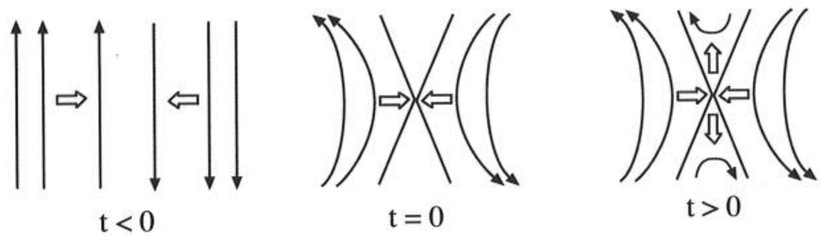
When collisions are not neglected, the plasma gets finite resistivity

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta \boldsymbol{j}$$

➔ the flux is not conserved,

➔ the field lines can detach from the plasma

➔ door is open to dramatic events, such as …

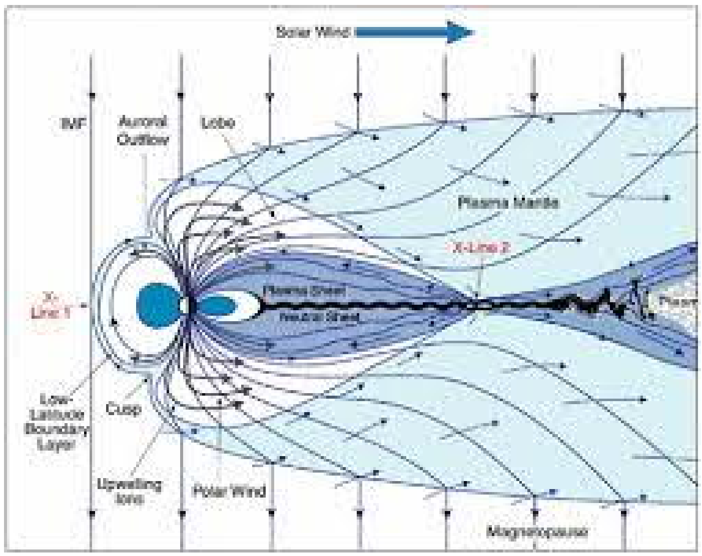


Magnetic reconnection !

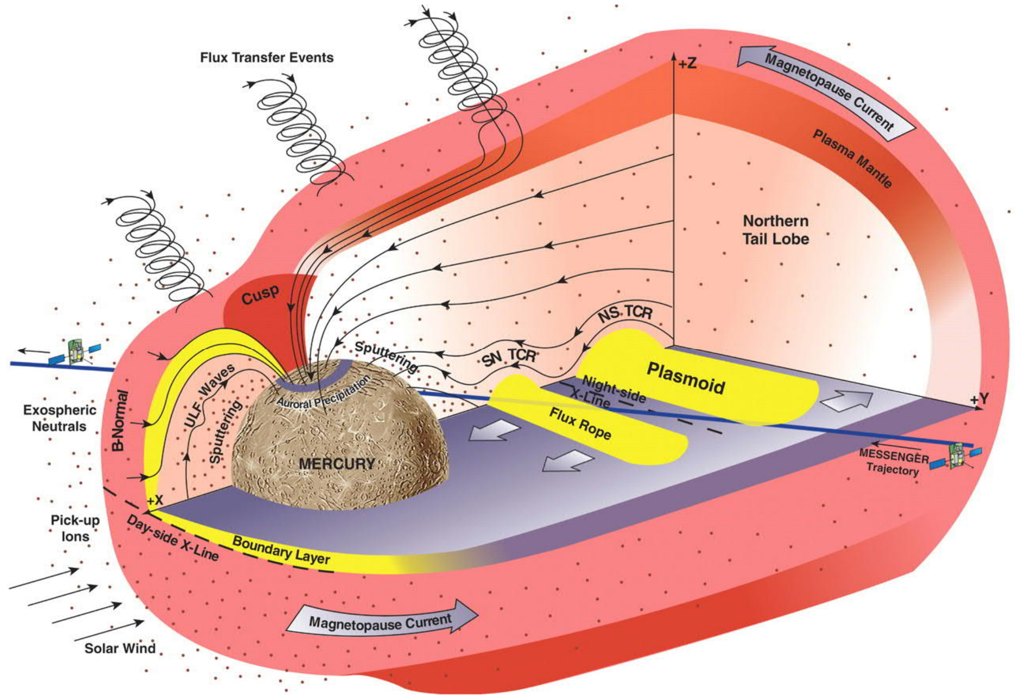
In a reconnection, magnetic field finds a new *equilibrium* with lower energy.

Is reconnection only an academic artefact of theory?

Reconnection is real !



Physics of magnetic reconnection in Earth's magnetotail



From: "*MESSENGER Observations of Magnetic Reconnection in Mercury's Magnetosphere*" *Science 01 May 2009*

But not fully understood: too fast for resistive time scales…

Is ideal MHD good for anything?

An important feature of plasma physics:

Plasma phenomena occur in a wide range of time and spatial scales.

For each scale, it is useful to have a theoretical ***model*** *that takes advantage of the scale.*

Or, in plain English

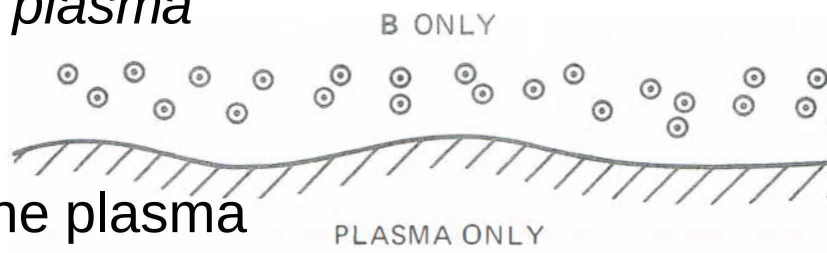
For each scale, it is wise to make approximations eliminating ***unnecessary*** *complications.*

Resistive time scale is long

➔ *ideal MHD is the right tool for 'fast' phenomena*

How long is resistive time scale?

Simpliest case: *magnetic field penetrating into a plasma*



$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta \boldsymbol{j}$ & $\dfrac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E}$

Take plasma at rest ↔ only B field moving into the plasma

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times (\eta \boldsymbol{j}) = \frac{\eta}{\mu_0} \nabla \times (\nabla \times \boldsymbol{B}) = \frac{\eta}{\mu_0} \nabla^2 \boldsymbol{B}$$

Identify the equation: we got the *diffusion equation* for B field!!

So physically the process can be considered as *diffusion of field lines!*

Separate variables, estimate scale length $\nabla \sim 1/L_B$

➔ $\boldsymbol{B} = \boldsymbol{B}_0 \exp(t/\tau)$, where $\tau \sim \mu_0 L_B^2 / \eta$

Decay of the magnetic field

Resistive time τ also gives the *decay time* of the magnetic field in a resistive plasma:

Recall: in plasmas/materials $\frac{dw_B}{dt} = \eta j^2$, where w_B is energy *density*

In resistive time τ the change in the magnetic field's energy thus is

$$\Delta w_B \sim \eta j^2 \tau \sim \eta \left[\frac{B}{\mu_0 L_B}\right]^2 \frac{\mu_0 L_B^2}{\eta} \sim \frac{B^2}{\mu_0}$$

But this ”equals” (~) the energy density of the magnetic field!

➔ τ also gives the rate at which magnetic field dissipates (into heat) in plasma