Mixed integer linear programming (MILP)

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Contents

- Mixed integer linear programming (MILP)
  - Formulation
  - Graphical representation
  - Sample energy models
  - Solving MILP models
- Dynamic LP/MILP modelling

LP and MILP modelling

- Linear Programming and Mixed Integer Linear Programming are most commonly used approaches for practical problems because
  - the modelling techniques are very versatile and flexible
  - efficient and reliable solvers exist for these problems
- Arbitrary convex optimization problems can be approximated with LP models
- Non-convex optimization problems can be approximated with MILP models

Non-convex optimization problems

- When the (minimized) objective function or some of the constraints are not convex, then the problem is **non-convex**
- A non-convex problem may have several local optima
  - In the general case it is not possible to know at beforehand which local optimum is the global optimum → necessary to explore them all
  - It can be difficult (and even impossible) to ensure that all local optima have been explored
  - In MILP problems it can be ensured!

Mixed integer linear programming (MILP) model

- A mixed integer linear programming problem is similar to an LP model, but some of the variables have integer domain:

  min (max) $\mathbf{cx}+\mathbf{dy}$

  s.t.

  $A\mathbf{x}+B\mathbf{y} \leq \mathbf{b}$

  $\mathbf{x} \geq 0$

  $y_i \in \{0,1\}$ (or some other finite range of integers)

- If all variables are integers, the problem is a (pure) integer linear programming (ILP) problem

Properties of MILP models

- Special case of non-convex problems
  - Optimum is always at a corner point of an LP model that is obtained by fixing the integer variables to some feasible values
- Let $\mathbf{y}^*$ = vector of 0/1 values
  - Then $\mathbf{dy}^*$ = constant and B$\mathbf{y}$ **=** constant vector
  - An LP model results

min (max) $\mathbf{cx}$ + constant

s.t.

A$\mathbf{x} \leq \mathbf{b}$ **–** B$\mathbf{y}$ = constant vector

$\mathbf{x} \geq 0$

Properties of MILP models

- Reliable (but not so efficient) solution algorithms exist
  - The Branch&Bound algorithm will enumerate explicitly or implicitly the different value combinations of integer variables
  - This reduces the MILP problem into **multiple LP problems**
- Finite non-convex problems can be approximated by MILP models with arbitrarily good accuracy
  - In principle a MILP model can always be solved
  - However, the resulting model may become large and **very slow** to solve
    - number of LP models to solve can be astronomical

How to define a MILP model?

1. Write down a verbal explanation of what is the goal or purpose of the model
   – E.g. to minimize costs or maximize profit from some specific operation or activity
2. Define the **decision variables** (and parameters)
   – Specify if they are real numbers or **binary** or **general integers**
   – Use as descriptive or generic names as you like: x1, x2, fuel, …
   – Give short description for them
   – Also specify the unit (MWh, GJ, €/kg, m3/s, …)
3. Define the **objective function** to minimize or maximize as a *linear function* of the variables
4. Define the **constraints** as *linear* inequality or equality constraints of the variables

Example of energy MILP modelling

- Biofuel power plant that can be shut down
  - Plant operation follows a linear characteristic in a range

$$x_{el} = x_{bio}/R - P_{loss}$$

$$x_{el}^{min} \le x_{el} \le x_{el}^{max}$$

x_{bio} = biofuel consumption

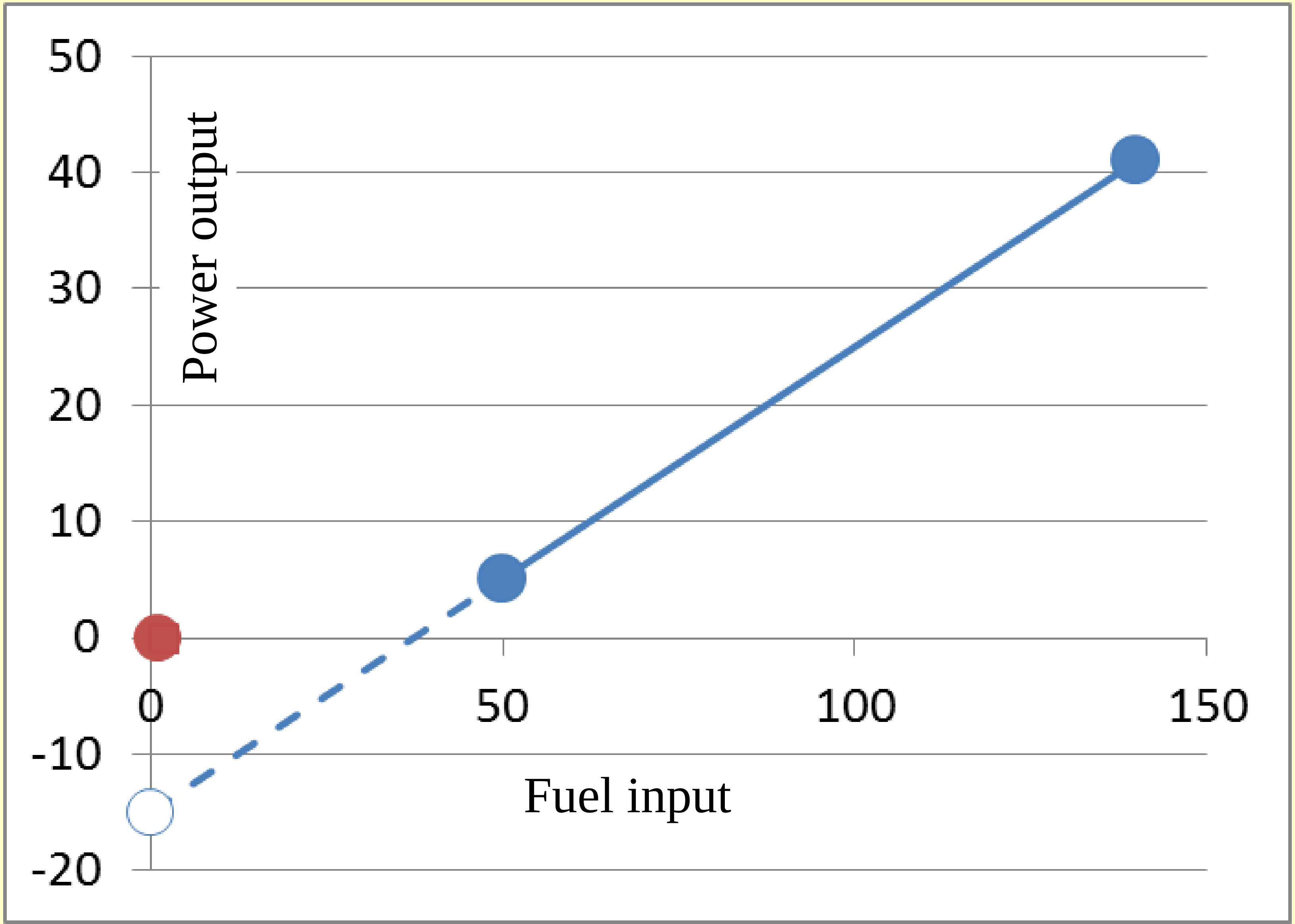
R = consumption ratio for biofuel

P_{loss} = constant loss

x_{el}^{min}, x_{el}^{max} = minimum/maximum power production

  - But when the plant is shut down, $x_{el} = x_{bio} = 0$

Biofuel power plant characteristic



Biofuel power plant MILP model

- A binary variable y is defined as a switch to determine if the plant is on (y=1) or off (y=0)
- Encoded model

$$\text{Max } c_{el}x_{el} - c_{bio}x_{bio}$$

$$x_{el} = x_{bio}/R - y^*P_{loss}$$

$$y^*x_{el}^{min} \le x_{el} \le y^*x_{el}^{max}$$

$$y \in \{0,1\}$$

c_{el} = price for sold power

c_{bio} = fuel price

  – The y-variable affects both the plant characteristic and bounds for power output

Wrong way to use binary variables

- Sometimes people try to write

$$\text{Max } y*(c_{el}x_{el} - c_{bio}x_{bio}) \qquad \text{// not linear}$$

$$x_{el} = y*(x_{bio}/R - P_{loss}) \qquad \text{// not linear}$$

$$x_{el}^{min} \leq x_{el} \leq x_{el}^{max} \qquad \text{// infeasible when plant is off}$$

$$y \in \{0,1\}$$

  - ♠ Objective is not linear, product of variables
  - ♠ Constraint is not linear, product of variables
  - ♠ Lower bound of x_{el} is infeasible when plant off
- Objective function and constraints in MILP model must be linear as in an LP model!

Encoding logical relations

- All logical operators ($\wedge, \vee, \neg, \ldots$) can be encoded using binary variables and linear constraints

  $X = Y \wedge Z \rightarrow x \le y;\ x \le z;\ x \ge y+z-1$

  $X = Y \vee Z \rightarrow x \le y+z;\ x \ge y;\ x \ge z$

  $X = \neg Y \rightarrow x = 1-y$

- Arbitrarily complex logical expressions can be encoded in sequence

  $Y = (Y_1 \wedge \neg Y_2) \vee Y_3 \Leftrightarrow Y = Z \vee Y_3;\ Z = Y_1 \wedge \neg Y_2$

  $\rightarrow y \le z+y_3;\ y \ge z;\ y \ge y_3;$

  $z \le y_1;\ z \le 1-y_2;\ z \ge y_1+(1-y_2)-1$

MILP-encoding of general non-convex problems

- A non-convex optimization problem is of form

$$\text{Min } f(x);\ \text{s.t. } x \in X$$

  – where f() is a non-convex function,

  – or X is a non-convex set,

  – or both

MILP-encoding of non-convex constraints

- X is partitioned into convex subsets $X = \cup X_i$
- A binary variable y_i is defined for each part
  - the part is enabled when $y_i=1$ and disabled when $y_i=0$
- Each subset is modelled by linear constraints

$$A_i x \leq b_i + \mathbf{M}(1-y_i)$$

- Binary variables activate exactly one set of constraints at a time

$$\Sigma y_i = 1$$
$$y_i \in \{0,1\}$$

MILP-encoding of non-convex objective function

- A non-convex objective function can be approximated by a piecewise linear function
- Example: min $f(x) = -x^2$ in range [-2,2]:
  - Choose points (X_i,F_i) along function
  - define (x,f) as convex combination of linear segments using **continuous variables x_i** and
  - **binary variables y_i** to enable exactly one segment

    $x=\Sigma_j x_i X_i$

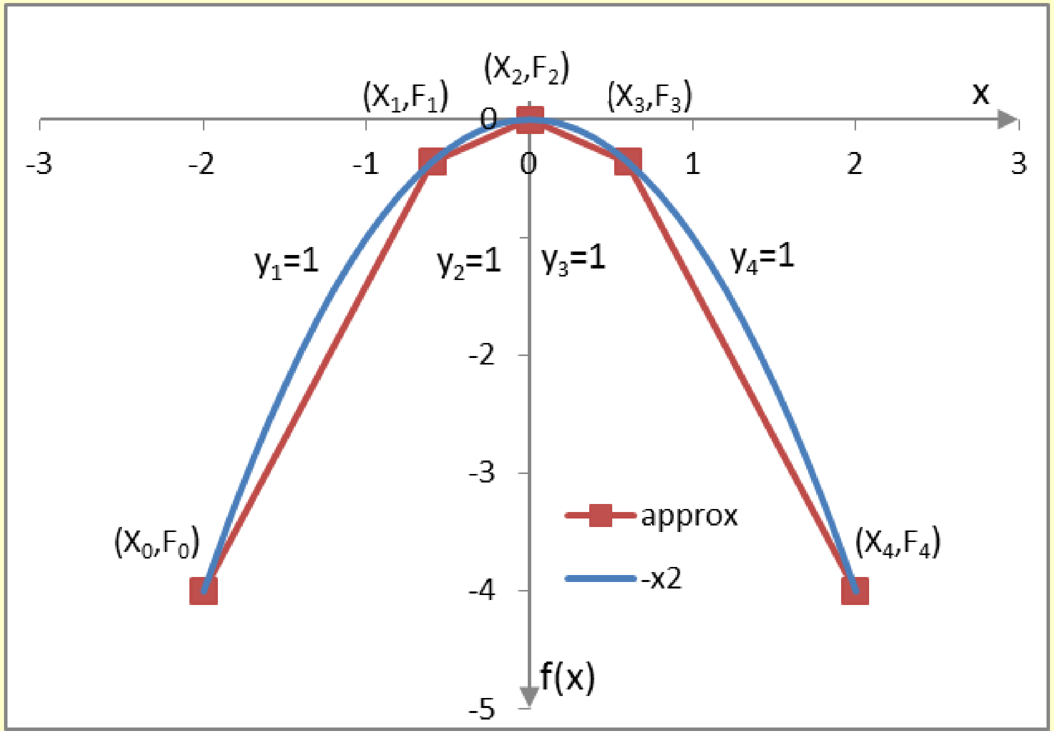
    $f=\Sigma x_i F_i$

    $\Sigma x_i=1$

    $x_i \leq y_i+y_{i+1}$

    $\Sigma y_i=1$

    $x_i \geq 0,\ y_i \in \{0,1\}$



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Convex CHP model

- The power plant characteristic defines in the P-Q plane the feasible operating area area
  - p = power production, q = heat production, c = fuel cost
- We encode the model as a **convex combination** of extreme (corner) points

max $cp - \Sigma_j c_j x_j$

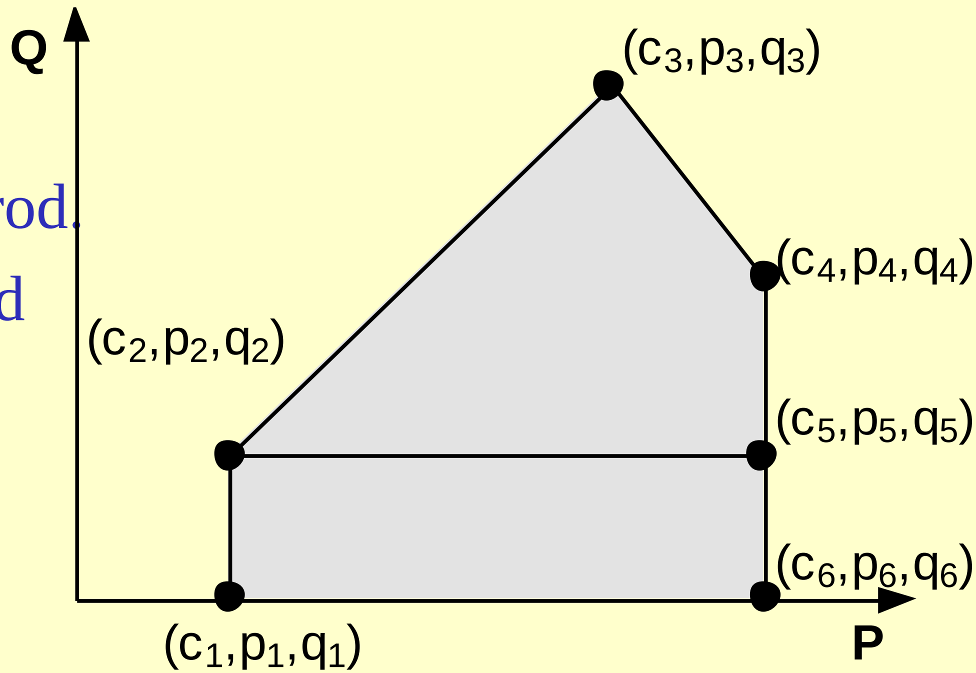
s.t.

$\Sigma_j p_j x_j = p$ // variable power prod.

$\Sigma_j q_j x_j = q$ // fixed heat demand

$\Sigma_j x_j = 1$ // convex comb.

$x_j \geq 0$



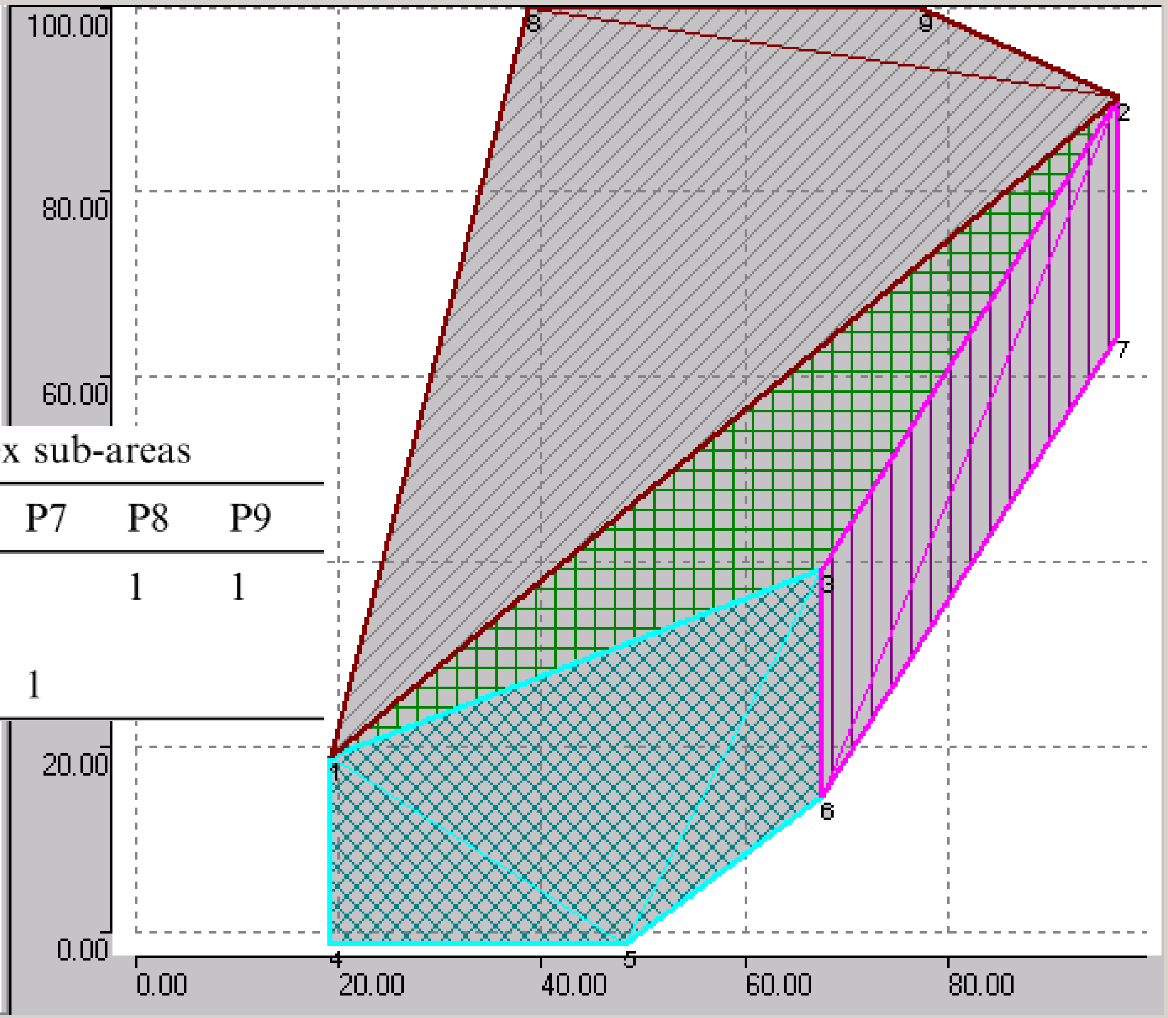
Non-convex CHP model

- Necessary when either (or both)
  - The cost function is non-convex
  - P-Q the characteristic is non-convex
    - E.g. when it is necessary to optimize the shut-down of the plant
- Idea
  - Partition objective function into convex parts
  - Partition characteristic into convex parts
  - Use 0/1 variables to choose in which area to operate

Sample non-convex cogeneration model

Characteristic Configuration

Order	P	Q	RP	RQ
1.	20	20	1.2	1.1
2.	100	90	1.2	1.1
3.	70	40	2	1.1
4.	20	0	3	1.1
5.	50	0	2.6	1.1
6.	70	15	2.7	1.1
7.	100	65	2.5	1.1
8.	40	100	1.2	1.01
9.	80	100	1.2	1.07



Allocation of characteristic points to convex sub-areas

Area	P1	P2	P3	P4	P5	P6	P7	P8	P9
A1	1	1	1					1	1
A2	1		1	1	1	1			
A3		1	1			1	1		

OK Cancel

R.

Non-convex cogeneration model

- Characteristic is partitioned in three convex parts

Allocation of characteristic points to convex sub-areas

Area	P1	P2	P3	P4	P5	P6	P7	P8	P9
A1	1	1	1					1	1
A2	1		1	1	1	1			
A3		1	1			1	1		

- A_j is **set of areas** to which x_j belongs
- Define zero-one variables y1, y2, y3, and allow exactly one of them to have value 1.
- y-variables select which corner points are allowed in the convex combination

$$x_j \leqslant \sum_{a \in A_j} y_a, \quad j \in J_u, \quad u \in U^*,$$

$$\sum_{a \in A_u} y_a = 1, \quad u \in U^*,$$

$$y_a \in \{0, 1\}, \quad a \in A_u, \quad u \in U^*.$$

Solving MILP models

- In principle it is possible to solve MILP problems using brute force:
  - Choose a value combination of integer variables
  - Solve the resulting LP problem
  - The best feasible solution among all combinations gives the optimum
- The number of problems to solve is exponential with respect to number of variables
  - With N binary variables, there are 2^N combinations
  - $N=10 \rightarrow 1024$, $20 \rightarrow 10^6$, $30 \rightarrow 10^9$, …

Solving MILP models

- The Branch & Bound algorithm solves MILP models more efficiently by solving only a small fraction of all combinations
  - Still solution time may be exponential
- Standard software
  - CPLEX, GAMS, Lindo, Lingo, Excel Solver …
- Very efficient specialized algorithms exist for the extreme point formulation
  - Power Simplex, Extended Power Simplex, Tri-Commodity Simplex, …

One specialized algorithm for CHP

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Non-convex power plant modelling in energy optimisation

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Dynamic systems

- A dynamic system is one which develops in time
  - Opposite: static system
- Normally, a dynamic system is modelled by discretizing it into a sequence static models that are connected by dynamic constraints

Dynamic energy models

- Examples:
  - Yearly CHP planning model is represented by a sequence of 8760 hourly models
    - Dynamic constraints result from
      - energy storages
      - startup and shutdown costs and restrictions
  - Daily hydro power scheduling is represented by a sequence of 96 15min models
    - Dynamic constraints result from water level/amount in reservoirs and waterflows between reservoirs

Dynamic optimization

- Different ways to model and solve dynamic systems exist
  - Multiperiod LP/MILP models
  - General mathematical optimization models
  - Dynamic programming algorithm
  - Other network algorithms

Multiperiod LP/MILP modelling

- A multiperiod LP/MILP model is an LP/MILP model with a special structure
  - Time horizon is divided into a sequence of time periods, t= 1, …, T.
  - The behaviour during each period t is modelled by a static LP/MILP model
  - The periods are connected by dynamic constraints linking together
    - subsequent period models pairwise, or
    - all period models for the entire horizon

Multiperiod LP modelling: Subsequent constraints for heat storage

- Let s[t] denote storage level at end of period t (= storage level at beginning of period t+1)
- During each period, s[t] depends on previous level plus charge minus discharge

  s[t] = EtaS*s[t-1] + EtaIn*sin[t] – sout[t]; (t= 1, …T)

  q[t] – sin[t] + EtaOut*sout[t] = qdemand[t];// heat balance

  - s[0] is the initial storage level (fixed, e.g. 0)
  - q[t] is production of heat in period t
  - qdemand[t] is the demand for heat in period t
  - EtaS = storage efficiency in time, 1 if no loss
  - EtaIn = efficiency of charging storage
  - EtaOut = efficiency of discharging storage

Multiperiod LP modelling: Subsequent ramp constraints

- Ramp constraints state that the plant may adjust during an hour the production too fast up (uramp) or down (dramp)
  - There may be ramp constraints for
    - power production p
    - heat production q
    - fuel consumption f (boiler operation)

$-Pdramp \leq p[t] - p[t-1] \leq Puramp;$

$-Qdramp \leq q[t] - q[t-1] \leq Quramp;$

$-Fdramp \leq f[t] - f[t-1] \leq Furamp;$

Multiperiod LP modelling: Constraints over planning horizon

- Emission limit for planning horizon
  - Sum(t:= 1 to T, e[t]) ≤ Emax;
- Fuel availability constraint for planning horizon
  - Sum(t:= 1 to T, f[t]) ≤ Fmax;
- Compute overall profit during planning horizon
  - Sum(t:= 1 to T, C[t]) = Ctotal;
    - Normally there is no constraint on overall profit, that is just something to be maximized

Multiperiod MILP modelling: Startup/shutdown costs and constraints

- On/off status of power plant is represented by binary variable y[t] and startup/shutdown by zup[t], zdown[t]

  zup[t] >= y[t] - y[t-1];

  zdown[t] >= y[t-1] - y[t]; (t= 1, …T)

  - only y[t] must be binary variables, zup&zdown can be real
  - y[0] is initial on/off status, which is fixed
- Startup/shutdown costs are included into objective

  Min …+ cup[t]*zup[t] + cdown[t]*zdown[t];
- Startup/shutdown restrictions are represented as logical constraints

  zup[t] <= 1-zdown[t-1]; // disable immediate startup

  zup[t] <= 1-zdown[t-2]; // and startup with 2-period delay etc.
- A long-term model can be large and too complex to solve