PHYS-C0252 - Quantum Mechanics Part 2
Section 1.3

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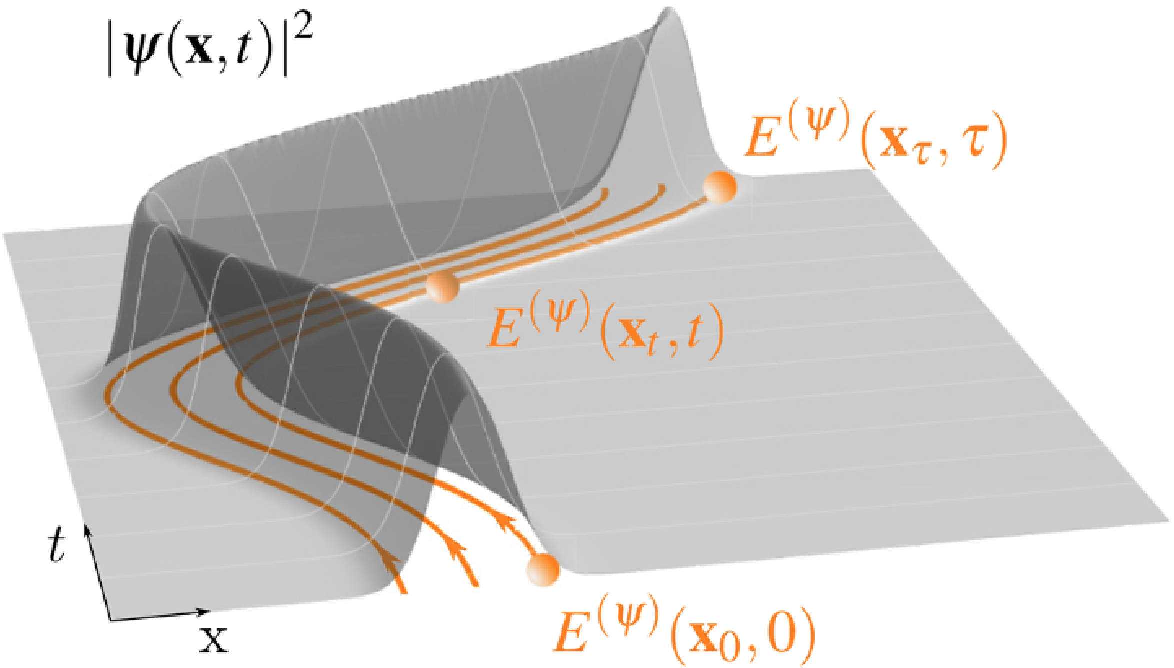

1.3 QHO in the position basis

- Another way of solving for the eigenfunctions and -values of the QHO is based on writing the Schrödinger equation in its natural position basis, where we define the *wave function* as $\psi(x) \equiv \langle x|\psi\rangle$
- This is a coordinate representation by using the basis set {| *x* >} of the position operator $\hat{q}$
- The Schrödinger equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + \frac{m\omega^2}{2}x^2\psi(x) = E\psi(x)$$

To simplify the equation, it is useful to define

$$q = \left(\frac{m\omega}{\hbar}\right)^{1/2} x; \;\; \lambda = \frac{2E}{\hbar\omega}; \quad \psi(x) = u(q)$$

which gives (check)

$$\frac{d^2u}{dq^2} + (\lambda - q^2)u = 0$$

This is an inhomogeneous but linear DE which can be solved in multiple ways. The easiest is to write *u*(*q*) as

$$u(q) = H(q)e^{-q^2/2}$$

where the functions (polynomials) $H(q)$ satisfy the DE

$$H'' - 2qH' + (\lambda - 1)H = 0$$

- The solutions of this DE are polynomial Hermite functions of order n that can be explicitly constructed by inserting a power law expansion to the DE (homework problem). This requires that $\lambda = 2n + 1$ which gives

$$E_n = \hbar\omega(n + \frac{1}{2})$$

- The Hermite polynomials can be generated through

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}$$

- The complete, normalized eigenfunctions of the QHO are given by

$$\psi_n(x) = \left(\frac{\alpha}{\sqrt{\pi} 2^n n!} \right) H_n(\alpha x) e^{-\alpha^2 x^2 / 2}$$

$$\alpha = \sqrt{m\omega/\hbar}$$

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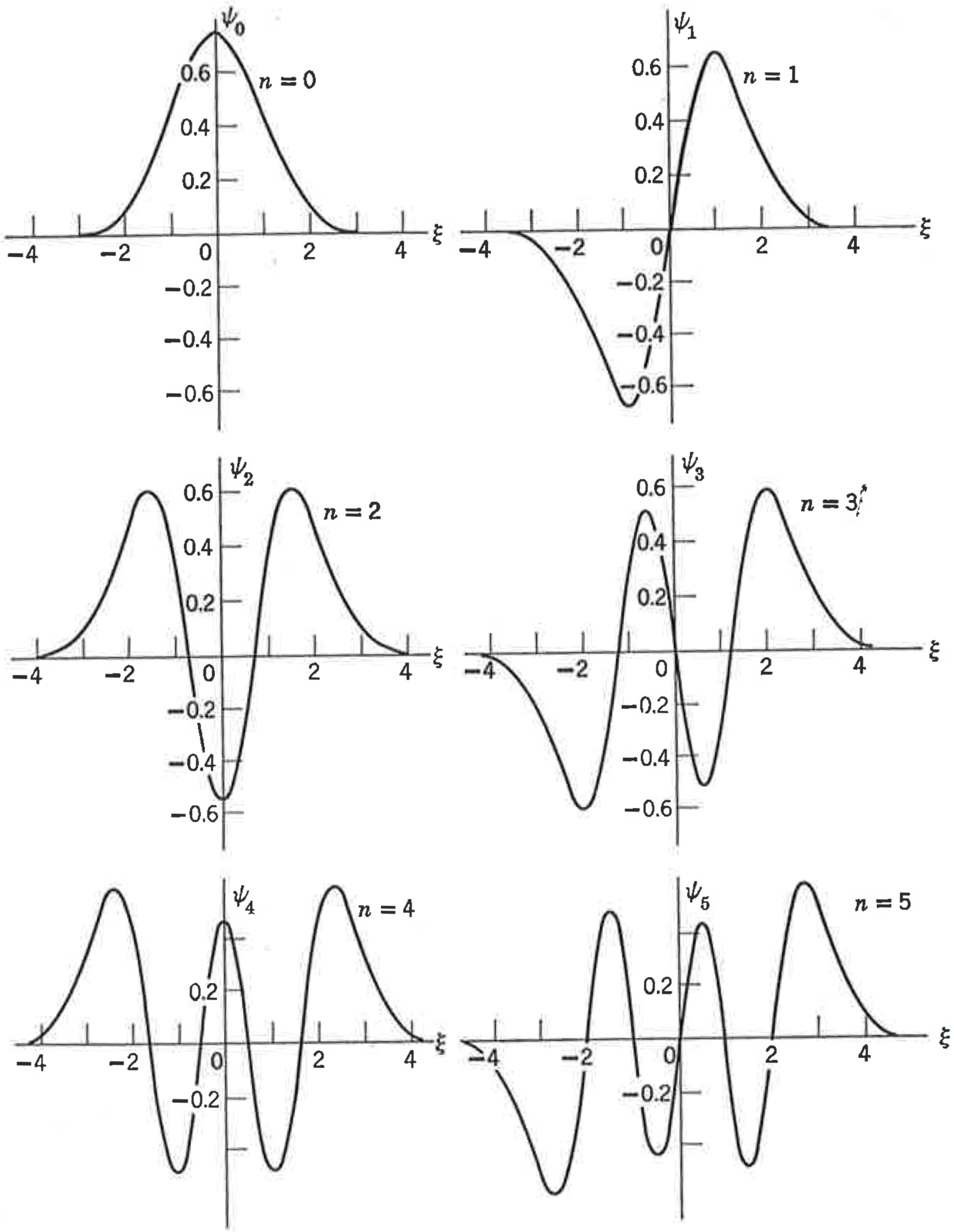
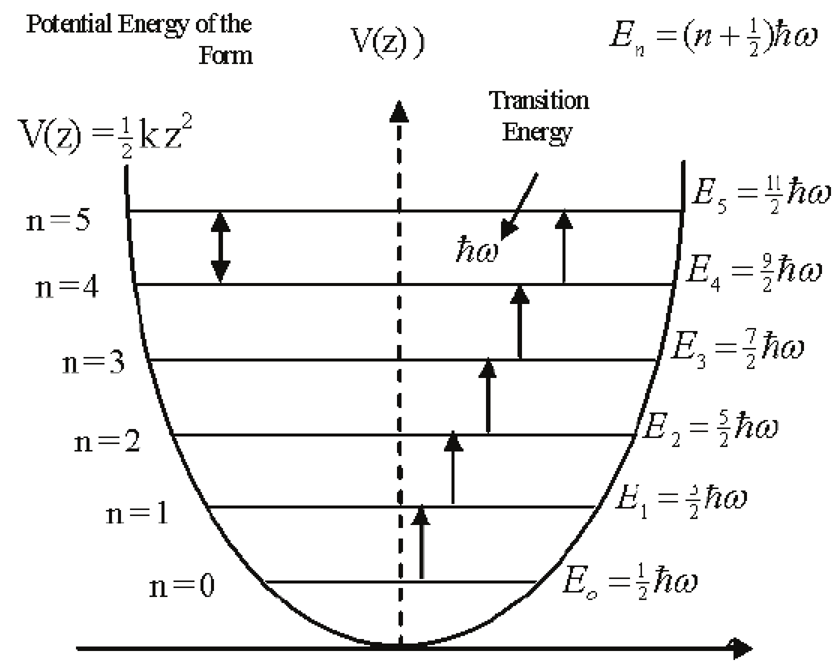
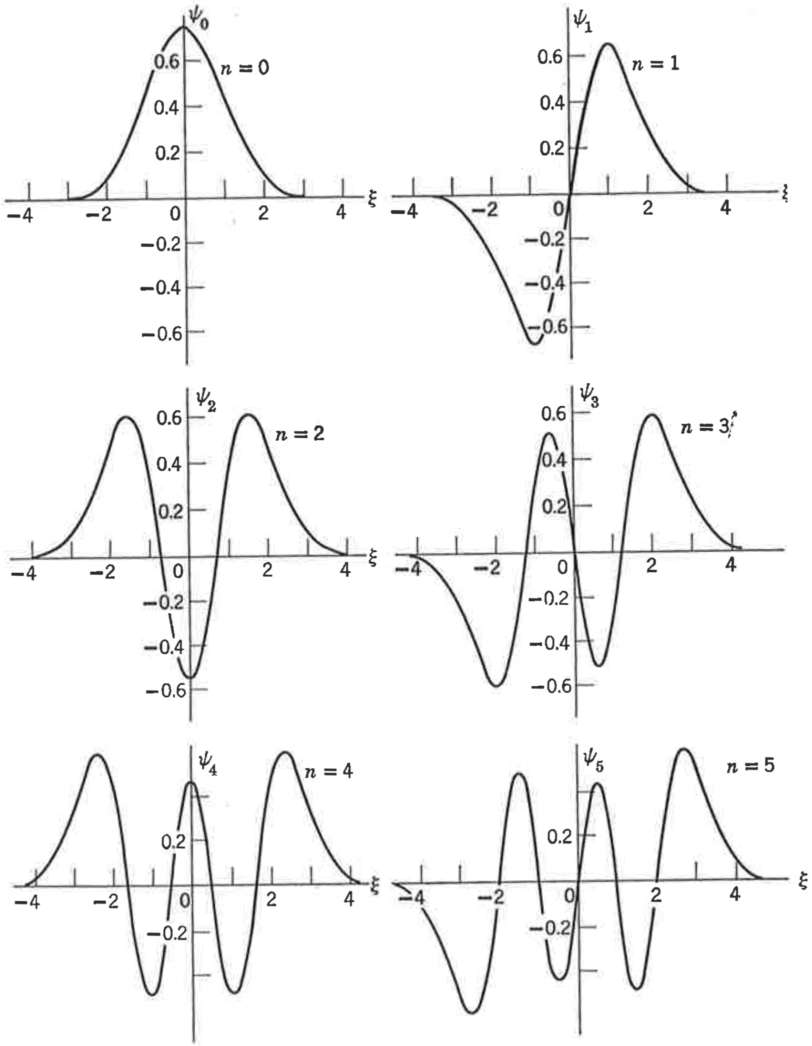
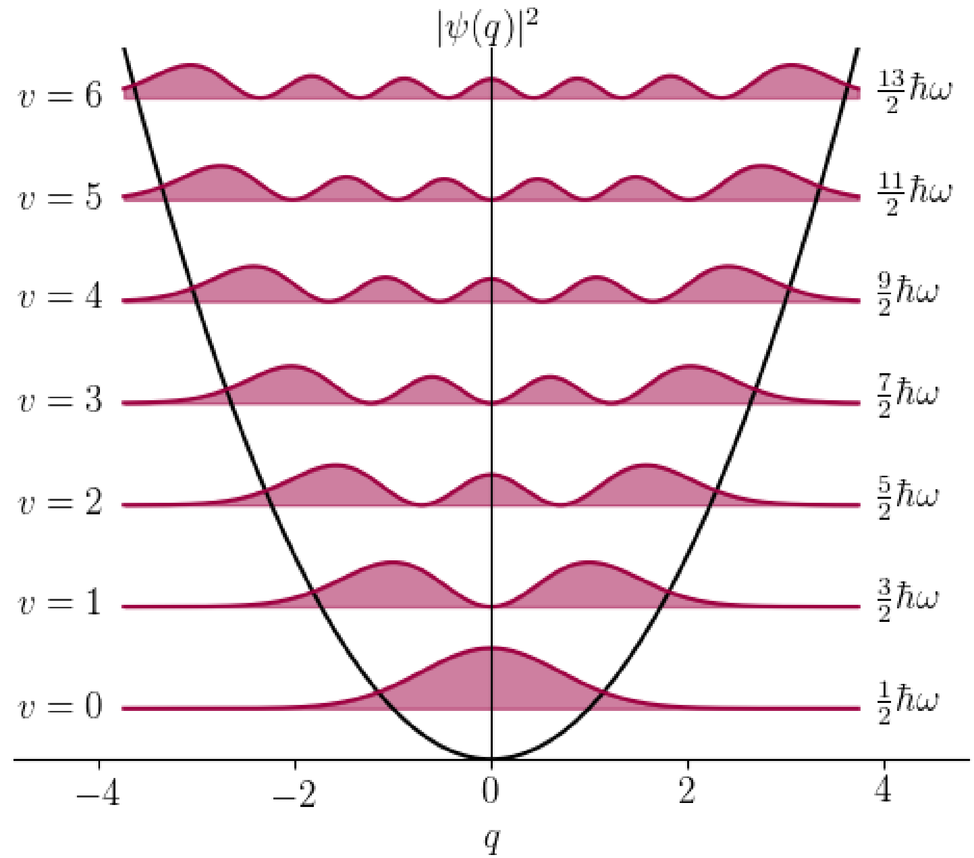


$$\xi = \sqrt{m\omega/\hbar x}$$



$$\psi_n(x) = \left(\frac{\alpha}{\sqrt{\pi}2^n n!}\right) H_n(\alpha x) e^{-\alpha^2 x^2/2} \qquad \xi = \sqrt{m\omega/\hbar}x$$





Probability (density) of finding the particle at any given point

- The importance of the Hermite functions is that they form a *complete, orthogonal set of eigenfunctions in the Hilbert space*, where the inner product is defined by

$$\int_{-\infty}^{\infty} H_n(\xi) H_k(\xi) e^{-\xi^2} d\xi = 0 \qquad \text{for} \quad n \neq k$$

The complete set of *orthonormal* eigenfunctions

$$\int_{-\infty}^{\infty} dx \psi_n^*(x) \psi_k(x) = \delta_{nk}$$

is given by

$$\psi_n(x) = 2^{-n/2} (n!)^{-1/2} \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-m\omega x^2/(2\hbar)} H_n\left(\sqrt{\frac{m\omega}{\hbar}} x\right)$$