PHYS-E055101 Low Temperature Physics: Nanoelectronics

Quantum amplifiers

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I. INTRODUCTION

In this lecture we study the phenomenon of amplification of signals. Amplifiers are very useful devices: they allow us to bring the signal of interest above the noise level, so it can be measured and recorded. Usually in mesoscopic physics one encounters rather standard room-temperature amplifiers as well as more specialized cryogenic (typically microwave-range) HEMT (high electron mobility transistors) amplifiers. As we will see, in the process of amplification, an amplifier would inevitably add its own noise to the signal. This is an unwanted process, and in this lecture we will investigate one way to reduce it, namely by the use of the process of parametric amplification. The applications of quantum amplifiers include the generation of entangled states, squeezing of the vacuum noise, and back-action evading measurements of position.

II. CLASSICAL PARAMETRIC EXCITATION

To develop an intuition about what is parametric amplification, it is useful to start with the classical case. Consider a simple system, such as a harmonic oscillator with bare oscillation frequency ω_0 and with a friction force proportional to the speed and characterized by a friction coefficient κ. Imagine now that we change the frequency of the oscillator, $\omega_0 \to \omega(t)$. The classical equation of motion is

$$\frac{d^2x(t)}{dt^2} + \kappa\frac{dx(t)}{dt} + \omega^2(t)x(t) = 0. \tag{1}$$

An elegant way of getting rid of the first-derivative friction term is to imagine that its role will be anyway to produce an exponential decay of the solution. This intuition is correct - let us make the substitution

$$x(t) = e^{-\frac{\kappa}{2}t}q(t). \tag{2}$$

One can immediately verify that it produces a new equation without the friction term but with a renormalized frequency for the harmonic oscillator,

$$\frac{d^2}{dt^2}q(t) + \left[\omega^2(t) - \frac{\kappa^2}{4}\right]q(t) = 0. \tag{3}$$

Let us define the time-independent renormalized frequency as

$$\omega_n^2 = \omega_0^2 - \frac{\kappa^2}{4}, \tag{4}$$

and we include the time-dependent part of $\omega(t)$ in a dimensionless function $f(t)$

$$\omega^2(t) = \omega_0^2 + \omega_n^2 f(t). \tag{5}$$

With these notations we can rewrite Eq. (3)

$$\frac{d^2}{dt^2}q(t) + \omega_n^2 \left[1 + f(t)\right] q(t) = 0. \tag{6}$$

Let us now consider a particular form for the function $f(t)$, namely $f(t) = f_0 \sin \omega_p t$, where ω_p is called **pump frequency**. As we will see, the important case is when $\omega_p/2$ is close to the natural oscillation frequency of the harmonic oscillator - in other words the oscillator is pumped at twice its natural frequency. In this case we will have the phenomenon of **parametric resonance**. We would like now to solve approximately Eq. (6). For this, let us separate the dynamics of $q(t)$ into a fast dynamics, with frequency ω_p and amplitudes $a(t)$ and $a^*(t)$ that are comparatively slow. We search for a solution

$$q(t) = a(t)e^{-i\frac{\omega_p}{2}t} + a^*(t)e^{i\frac{\omega_p}{2}t}. \tag{7}$$

Let us plug this in Eq. (6). Because $a(t)$ and $a^*(t)$ are assumed slow, we can neglect their second-order derivatives. Of course, this has to be justified self-consistently (the solutions that we get for $a(t)$ and $a^*(t)$ should indeed be slow compared to $\omega_p/2$). Also we neglect fast-oscillating terms at frequencies $\pm 3\omega_p/2$, which is similar to doing a rotating wave approximation. With this, we get

$$\omega_p \frac{d}{dt}a(t) = \frac{f_0}{2}\omega_n^2 a^*(t) + i\left[\left(\frac{\omega_p}{2}\right)^2 - \omega_n^2\right] a(t), \tag{8}$$
$$\omega_p \frac{d}{dt}a^*(t) = \frac{f_0}{2}\omega_n^2 a(t) - i\left[\left(\frac{\omega_p}{2}\right)^2 - \omega_n^2\right] a^*(t). \tag{9}$$

Next, let us make the substitution

$$a(t) = r(t)e^{i\theta(t)}, \tag{10}$$

which, when inserted in Eq. (8), produces

$$\dot{r}\cos\theta - r\dot{\theta}\sin\theta = \frac{f_0}{2\omega_p}\omega_n^2 r\cos\theta - \frac{(\omega_p/2)^2 - \omega_n^2}{\omega_p} r\sin\theta, \tag{11}$$
$$-\dot{r}\sin\theta - r\dot{\theta}\cos\theta = \frac{f_0}{2\omega_p}\omega_n^2 r\sin\theta - \frac{(\omega_p/2)^2 - \omega_n^2}{\omega_p} r\cos\theta. \tag{12}$$

Next, we multiply the first equation by $\cos\theta$ and the second by $\sin\theta$ and substract them; then we multiply the first by $\sin\theta$ and the second by $\cos\theta$ and add them. We find

$$\frac{dr}{dt} = \frac{f_0\omega_n^2}{2\omega_p} r\cos\theta, \tag{13}$$

$$\frac{d\theta}{dt} = -\frac{f_0\omega_n^2}{2\omega_p}(\sin 2\theta - \sin 2\theta_\infty), \tag{14}$$

where

$$\sin 2\theta_\infty = \frac{2}{f_0\omega_n^2}\left[\left(\frac{\omega_p}{2}\right)^2 - \omega_n^2\right]. \tag{15}$$

Now, let us look at Eq. (14). We are interested in the long-term (asymptotic) behavior of the solution. Let us consider a small difference between θ and θ_∞; then we linearize

$$\sin 2\theta - \sin 2\theta_\infty = 2\cos(\theta+\theta_\infty)\sin(\theta-\theta_\infty) \approx 2(\theta-\theta_\infty)\cos 2\theta_\infty. \tag{16}$$

Then the solution of Eq. (14) is

$$\theta(t) = \theta_\infty + [\theta(0) - \theta_\infty]\exp\left[-\frac{f_0\omega_n^2}{\omega_p}(\cos 2\theta_\infty)t\right]. \tag{17}$$

Clearly the phase evolves asymptotically towards the value θ_∞ given by Eq. (15), a phenomenon called **phase locking**.

Next, we solve Eq. (13) with the phase locked to θ_∞. The result is

$$r(t) = r(0)\exp\left[\frac{f_0\omega_n^2}{2\omega_p}(\cos 2\theta_\infty)t\right]. \tag{18}$$

Here the initial values of r and θ are set by $x(0) = 2r(0)\cos\theta(0)$. In terms of the original variable x, we get

$$x(t) = 2r(0)\exp\left[\left(\frac{f_0\omega_n^2}{2\omega_p}\cos 2\theta_\infty - \frac{\kappa}{2}\right)t\right]\cos\left[\theta(t) + \frac{\omega_p}{2}t\right]. \tag{19}$$

This shows that there exists a regime where $x(t)$ increases exponentially with time, if

$$\frac{f_0\omega_n^2}{2\omega_p}\cos 2\theta_\infty - \frac{\kappa}{2} > 0 \tag{20}$$

is satisfied. This results in a **parametric instability** of the system. Obviously the maximum growth rate happens at resonance $\omega_p/2 = \omega_n$ (resulting in $\theta_\infty = 0$) which is the condition for **parametric resonance**.

One wonders if this effect can be used to amplify signals. Indeed, if we start with a certain value $x(0)$ for a signal, then by applying the parametric modulation the signal will grow very fast. This is what we will do next, in the quantum case and using the input-output formalism. As you will see, this is a natural application of the input-output theory: after all, real amplifiers are devices with an input and an output.

III. QUANTUM AMPLIFIERS

A. General theory of quantum amplification

We start with a general classification of quantum amplifiers. Let us introduce as usual the symmetrized noise: for an operator $\hat{O}$ we define

$$\left(\Delta\hat{O}\right)^2 = \frac{1}{2}\left\langle\left\{\hat{O},\hat{O}^\dagger\right\}\right\rangle - \left|\left\langle\hat{O}\right\rangle\right|^2. \tag{21}$$

Before this, we will introduce the so-called **field quadratures** $\hat{x}$ and $\hat{y}$, that can be defined for any field $\hat{a}$,

$$\hat{x} = \frac{1}{\sqrt{2}}\left[\hat{a} + \hat{a}^\dagger\right], \tag{22}$$

$$\hat{y} = \frac{1}{\sqrt{2}i}\left[\hat{a} - \hat{a}^\dagger\right]. \tag{23}$$

For example, suppose that we have the voltage across a transmission line or inside a coplanar waveguide resonator (a finite-length section of a transmission line), then

$$\hat{V}(t) = \hat{x}\cos\omega t + \hat{y}\sin\omega t, \tag{24}$$

$$= \frac{1}{\sqrt{2}}\left(\hat{a}e^{-i\omega t} + \hat{a}^\dagger e^{i\omega t}\right). \tag{25}$$

The Heisenberg uncertainty relations corresponding to the commutation relations

$$[\hat{x},\hat{y}] = i, \tag{26}$$

are

$$\Delta x\Delta y \geq \frac{1}{2}. \tag{27}$$

A mode is then called **squeezed** if the uncertainty in one of the quadratures is less than $1/\sqrt{2}$, for example

$$\Delta x < 1/\sqrt{2}. \tag{28}$$

Inevitably, due to the uncertainty relations, the other mode will be larger than 1/2

$$\Delta y > 1/\sqrt{2}. \tag{29}$$

With these definitions, and if G is gain of the amplifier, we can classify the amplifiers into two classes:

- **Phase-insensitive (phase-preserving) amplifiers**

  Main features:

  [1] Both input quadratures are multiplied by $\sqrt{G}$ - no matter what the phase of the signal (this is why it is called phase-insensitive).

  [2] Must add at least 1/2 of quantum noise (when referred to the input) in the amplified signal in order to satisfy the commutation relations.

- **Phase-sensitive amplifiers**

  Main features:

  [1] The quadratures are transformed by different amounts: one is multiplied by $\sqrt{G}$, while the other one is mutiplied by $1/\sqrt{G}$.

  [2] It does not add noise (noiseless amplification) in the output mode because it satisfies the commutation relations.

B. Phase-insensitive amplifiers and the Haus-Caves theorem

The question we would like to answer is why can't we amplify both quadratures? Note that this is already forbidden in classical physics: Liouville's theorem tells us that for a single degree of freedom the volume in phase space is constant. The only way out is to introduce another degree of freedom, called *idler*, which allows us to bypass Liouville's theorem.

In other words, suppose we call the output amplified mode $\hat{b}$ and we denote by $\hat{a}$ the mode to be amplified. Then we would like to have

$$\hat{b} = \sqrt{G}\hat{a}, \tag{30}$$

but this will contradict the commutation relations $[\hat{a}, \hat{a}^{\dagger}] = 1$ and $[\hat{b}, \hat{b}^{\dagger}] = 1$. We are therefore forced to write

$$\hat{b} = \sqrt{G}\hat{a} + \hat{\mathcal{F}}. \tag{31}$$

Now an important difference between classical and quantum physics comes into play. To reduce the noise of the field $\mathcal{F}$ we can cool this mode to a low enough temperature. Classically, the noise would be reduced to zero, since the thermal noise at zero temperature is zero. However, quantum physics limits the reduction of the noise to a value that is due to vacuum fluctuations. This intuition is captured in the following theorem due to Haus and Caves.

The Haus-Caves theorem

The theorem says that the minimum value of the noise at the output of a phase-preserving amplifier is given by the input noise multiplied by the gain plus the noise from the idler,

$$(\Delta\hat{b})^2 \geq G(\Delta\hat{a})^2 + \frac{1}{2}|G-1|. \tag{32}$$

Proof: We have

$$\begin{aligned}
(\Delta\hat{b})^2 &\equiv \frac{1}{2}\left\langle\left\{\hat{b},\hat{b}^\dagger\right\}\right\rangle - \left|\left\langle\hat{b}\right\rangle\right|^2 & (33)\\
&= \frac{1}{2}\left\langle(\sqrt{G}\hat{a}+\hat{\mathcal{F}})(\sqrt{G}\hat{a}^\dagger+\hat{\mathcal{F}}^\dagger)+(\sqrt{G}\hat{a}^\dagger+\hat{\mathcal{F}}^\dagger)(\sqrt{G}\hat{a}+\hat{\mathcal{F}})\right\rangle - & (34)\\
&-(\sqrt{G}<\hat{a}>+<\hat{\mathcal{F}}>)(\sqrt{G}<\hat{a}^\dagger>+<\hat{\mathcal{F}}^\dagger>). & (35)
\end{aligned}$$

Next, we use the fact that the idler field is a distinct uncorrelated with the input signal, that is

$$\left[\hat{a},\hat{\mathcal{F}}\right]=0, \qquad \left[\hat{a},\hat{\mathcal{F}}^\dagger\right]=0, \tag{36}$$

and

$$\left\langle\hat{a}\hat{\mathcal{F}}\right\rangle=0, \qquad \left\langle\hat{a}\hat{\mathcal{F}}^\dagger\right\rangle=0. \tag{37}$$

As a result,

$$\begin{aligned}
(\Delta\hat{b})^2 &= G(\Delta\hat{a})^2 + \frac{1}{2}\left\langle\left\{\hat{\mathcal{F}},\hat{\mathcal{F}}^\dagger\right\}\right\rangle & (38)\\
&\geq G(\Delta\hat{a})^2 + \frac{1}{2}\left|\left\langle\left[\hat{\mathcal{F}},\hat{\mathcal{F}}^\dagger\right]\right\rangle\right|. & (39)
\end{aligned}$$

Now the commutator can be easily evaluated,

$$\begin{aligned}
1 = [\hat{b},\hat{b}^\dagger] = [\sqrt{G}\hat{a}+\hat{\mathcal{F}},\sqrt{G}\hat{a}^\dagger+\hat{\mathcal{F}}^\dagger] &= G[\hat{a},\hat{a}^\dagger]+[\hat{\mathcal{F}},\hat{\mathcal{F}}^\dagger] = & (40)\\
&= G+[\hat{\mathcal{F}},\hat{\mathcal{F}}^\dagger], & (41)
\end{aligned}$$

therefore

$$[\hat{\mathcal{F}},\hat{\mathcal{F}}^\dagger] = 1-G. \tag{42}$$

As a result, we get the Haus-Caves theorem in the form

$$(\Delta\hat{b})^2 \geq G(\Delta\hat{a})^2 + \frac{1}{2}(G-1), \tag{43}$$

or equivalently

$$\frac{(\Delta\hat{b})^2}{G} \geq (\Delta\hat{a})^2 + \frac{1}{2G}(G-1). \tag{44}$$

Let us analyze this result in two limit cases:

- **No amplification** therefore $G = 1$, which means

  $$(\Delta\hat{b})^2 = (\Delta\hat{a})^2, \tag{45}$$

  therefore no noise is added, $\frac{1}{2G}(G-1) = 0$.

- **Large amplification**, meaning $G \gg 1$, resulting in

  $$\frac{(\Delta\hat{b})^2}{G} \geq (\Delta\hat{a})^2 + \frac{1}{2}. \tag{46}$$

  This shows that, for any reasonably good amplifier ($G \gg 1$ is not so difficult to realize!) there is a minimum amount of noise, equal to $1/2$, which is inevitably added to the input signal. One says that the noise added by the amplifier, when referred to the input, is at least $1/2$.

Note: Eq. (39) can be proved in the following way. Let $\hat{\mathcal{F}} = \hat{\mathcal{F}}_1 + i\hat{\mathcal{F}}_2$, therefore

$$\hat{\mathcal{F}}_1 = \frac{1}{2}(\hat{\mathcal{F}} + \hat{\mathcal{F}}^\dagger), \tag{47}$$

$$\hat{\mathcal{F}}_2 = \frac{1}{2i}(\hat{\mathcal{F}} - \hat{\mathcal{F}}^\dagger). \tag{48}$$

Some useful relations:

$$\hat{\mathcal{F}}^2 = \hat{\mathcal{F}}_1^2 - \hat{\mathcal{F}}_2^2 + i(\hat{\mathcal{F}}_1\hat{\mathcal{F}}_2 + \hat{\mathcal{F}}_2\hat{\mathcal{F}}_1), \tag{49}$$

$$\frac{1}{2}(\hat{\mathcal{F}}\hat{\mathcal{F}}^\dagger + \hat{\mathcal{F}}^\dagger\hat{\mathcal{F}}) = \hat{\mathcal{F}}_1^2 + \hat{\mathcal{F}}_2^2, \tag{50}$$

$$[\hat{\mathcal{F}}, \hat{\mathcal{F}}^\dagger] = -2i[\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2]. \tag{51}$$

Then

$$(\Delta\hat{\mathcal{F}})^2 = (\Delta\hat{\mathcal{F}}_1)^2 + (\Delta\hat{\mathcal{F}}_2)^2 \tag{52}$$

$$\geq 2(\Delta\hat{\mathcal{F}}_1)(\Delta\hat{\mathcal{F}}_2) \tag{53}$$

$$\geq \left|\left\langle\left[\hat{\mathcal{F}}_1, \hat{\mathcal{F}}_2\right]\right\rangle\right| \tag{54}$$

$$= \frac{1}{2}\left|\left\langle\left[\hat{\mathcal{F}}, \hat{\mathcal{F}}^\dagger\right]\right\rangle\right|. \tag{55}$$

For obtaining the inequality Eq. (54) we used the uncertainty principle applied to $\hat{\mathcal{F}}_1$, and $\hat{\mathcal{F}}_2$. The uncertainty principle in the Robertson formulation, which states that for two Hermitean operators $\hat{A}$ and $\hat{B}$ we have the inequality

$$(\Delta\hat{A})\cdot(\Delta\hat{B}) \geq \frac{1}{2}\left|\left\langle\left[\hat{A},\hat{B}\right]\right\rangle\right|. \tag{56}$$

IV. STANDARD MODELS FOR QUANTUM AMPLIFIERS

A. One-mode idler

In this case

$$\hat{\mathcal{F}} = \sqrt{G-1}\hat{d}^\dagger, \tag{57}$$

$$\hat{\mathcal{F}}^\dagger = \sqrt{G-1}\hat{d}, \tag{58}$$

where $\hat{d}$ is a bosonic mode. This choice results in the correct commutation relations for the idler,

$$[\hat{\mathcal{F}},\hat{\mathcal{F}}^\dagger] = (G-1)[\hat{d}^\dagger,\hat{d}] = 1-G. \tag{59}$$

In this case we have for the added noise for the idler,

$$\left\langle\{\hat{\mathcal{F}},\hat{\mathcal{F}}^\dagger\}\right\rangle = (G-1)\left\langle\{\hat{d},\hat{d}^\dagger\}\right\rangle = (G-1)\left(1+2\left\langle\hat{d}^\dagger\hat{d}\right\rangle\right) = G-1, \tag{60}$$

assuming that the idler mode $\hat{d}$ is assumed cooled to vacuum. As a result,

$$(\Delta\hat{b})^2 = G(\Delta\hat{a})^2 + \frac{1}{2}(G-1). \tag{61}$$

The main result is here that for a single-mode idler the Haus-Caves inequality becomes equality.

B. Two-mode idler

In this case we choose

$$\hat{\mathcal{F}} = \sqrt{G-1}(\cosh\theta\hat{d}_1^\dagger + \sinh\theta\hat{d}_2), \tag{62}$$

$$\hat{\mathcal{F}}^\dagger = \sqrt{G-1}(\cosh\theta\hat{d}_1 + \sinh\theta\hat{d}_2^\dagger). \tag{63}$$

The commutation relations of the idler are satisfied,

$$\left[\hat{\mathcal{F}}, \hat{\mathcal{F}}^\dagger\right] = (G-1)\left([\hat{d}_1^\dagger, \hat{d}_1]\cosh^2\theta + [\hat{d}_2, \hat{d}_2^\dagger]\sinh^2\theta\right) = 1 - G. \tag{64}$$

For the noise we obtain

$$\left\langle \{\hat{\mathcal{F}}, \hat{\mathcal{F}}^\dagger\}\right\rangle = (G-1)\left(\cosh^2\theta\left\langle\{\hat{d}_1^\dagger, \hat{d}_1\}\right\rangle + \sinh^2\theta\left\langle\{\hat{d}_2^\dagger, \hat{d}_2\}\right\rangle\right) \tag{65}$$

$$= (G-1)\left(\cosh^2\theta + \sinh^2\theta\right) = (G-1)(1+2\sinh^2\theta] \geq G-1, \tag{66}$$

and we have equality for $\theta = 0$ - in which case we are back to the case of a single-mode idler, see Eqs. (62, 63).

V. A NONLINEAR CAVITY AS A MODEL FOR PARAMETRIC AMPLIFICATION

Suppose we have a cavity filled with a nonlinear medium. This medium can be for example a second-order nonlinear susceptibility (denoted by $\chi^{(2)}$ in quantum optics). In microwave electronics, this nonlinearity can be realized by using Josephson junctions. Suppose then that we pump this cavity with a tone ω_p. In the lowest-order approximation, we can write the Hamiltonian

$$\hat{H} = \hbar\omega_c\hat{a}^\dagger\hat{a} - \frac{\hbar}{2}\left[\alpha^* e^{i\omega_p t} + \alpha e^{-i\omega_p t}\right](\hat{a}^\dagger + \hat{a})^2. \tag{67}$$

To get rid of the time-dependence in the exponentials, we move to a rotating-frame defined by the following transformation

$$\hat{U} = e^{i\frac{\omega_p}{2}\hat{a}^\dagger\hat{a}t}. \tag{68}$$

The following relations can be proved by a Taylor expansion of the exponentials,

$$e^{i\nu\hat{a}^\dagger\hat{a}t}\hat{a}e^{-i\nu\hat{a}^\dagger\hat{a}t} = e^{-i\nu t}\hat{a}, \tag{69}$$

$$e^{i\nu\hat{a}^\dagger\hat{a}t}\hat{a}^\dagger e^{-i\nu\hat{a}^\dagger\hat{a}t} = e^{i\nu t}\hat{a}^\dagger. \tag{70}$$

The transformed Hamiltonian is

$$\hat{\tilde{H}} = \left(i\hbar\frac{d\hat{U}}{dt}\right)\hat{U}^\dagger + \hat{U}\hat{H}\hat{U}^\dagger. \tag{71}$$

Let us take $\omega_p = 2\omega_c$ and then perform a rotating wave approximation (RWA),

$$\hat{\tilde{H}}_{\mathrm{RWA}} = -\frac{\hbar}{2}\left[\alpha^*\hat{a}^2 + \alpha\hat{a}^{\dagger 2}\right]. \tag{72}$$

Then, we can write the Heisenberg-Langevin equations

$$\frac{d}{dt}\hat{a} = \frac{i}{\hbar}\left[\hat{\tilde{H}}_{\mathrm{RWA}}, \hat{a}\right] - \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\hat{b}_{\mathrm{in}}, \tag{73}$$

or

$$\frac{d}{dt}\hat{a} = i\alpha\hat{a}^\dagger - \frac{\kappa}{2}\hat{a} - \sqrt{\kappa}\hat{b}_{\mathrm{in}} \tag{74}$$

$$\frac{d}{dt}\hat{a}^\dagger = -i\alpha^*\hat{a} - \frac{\kappa}{2}\hat{a}^\dagger - \sqrt{\kappa}\hat{b}_{\mathrm{in}}^\dagger \tag{75}$$

Now, taking the Fourier transform we find

$$\begin{pmatrix} \chi^{-1}(\omega) & -i\alpha \\ i\alpha^* & \chi^{-1}(\omega) \end{pmatrix} \begin{pmatrix} \hat{a}[\omega] \\ \hat{a}^\dagger[\omega] \end{pmatrix} = -\sqrt{\kappa} \begin{pmatrix} \hat{b}_{\mathrm{in}}[\omega] \\ \hat{b}_{\mathrm{in}}^\dagger[\omega] \end{pmatrix}, \tag{76}$$

where $\chi(\omega)$ is the response function of the cavity,

$$\chi(\omega) = \frac{1}{\frac{\kappa}{2} - i\omega}. \tag{77}$$

Note that at resonance $\omega = 0$ the system becomes unstable if $|\alpha|^2 \geq \kappa/2$. This equation can be readily solved,

$$\hat{a}[\omega] = \frac{-\sqrt{\kappa}\chi^{-1}(\omega)}{\chi^{-1}(\omega)^2 - |\alpha|^2}\hat{b}_{\mathrm{in}}[\omega] - \frac{i\alpha\sqrt{\kappa}}{\chi^{-1}(\omega)^2 - |\alpha|^2}\hat{b}_{\mathrm{in}}^\dagger[\omega], \tag{78}$$

$$\hat{a}^\dagger[\omega] = \frac{i\alpha^*\sqrt{\kappa}}{\chi^{-1}(\omega)^2 - |\alpha|^2}\hat{b}_{\mathrm{in}}[\omega] - \frac{\sqrt{\kappa}\chi^{-1}(\omega)}{\chi^{-1}(\omega)^2 - |\alpha|^2}\hat{b}_{\mathrm{in}}^\dagger[\omega]. \tag{79}$$

Now using the input-output relations

$$\hat{b}_{\mathrm{out}}[\omega] = \hat{b}_{\mathrm{in}}[\omega] + \sqrt{\kappa}\hat{a}[\omega], \tag{80}$$

we obtain the structure

$$\hat{b}_{\mathrm{out}}[\omega] = M(\omega)\hat{b}_{\mathrm{in}}[\omega] + L(\omega)\hat{b}_{\mathrm{in}}^\dagger[\omega], \tag{81}$$

$$\hat{b}_{\mathrm{out}}^\dagger[\omega] = \left(\hat{b}_{\mathrm{out}}[-\omega]\right)^\dagger = M(-\omega)^*\hat{b}_{\mathrm{in}}^\dagger[\omega] + L(-\omega)^*\hat{b}_{\mathrm{in}}[\omega], \tag{82}$$

where

$$M(\omega) = -\frac{(\kappa/2)^2 + \omega^2 + |\alpha|^2}{(\kappa/2 - i\omega)^2 - |\alpha|^2}, \tag{83}$$

$$L(\omega) = -\frac{i\alpha\kappa}{(\kappa/2 - i\omega)^2 - |\alpha|^2}. \tag{84}$$

It is now easy to verify that

$$|M(\omega)|^2 - |L(\omega)|^2 = 1. \tag{85}$$

This means that we can identify the gain as

$$G(\omega) = |M(\omega)|^2 . \tag{86}$$

We can also ignore the phases from the definition of M and L: for example, we can include the i into α and then being left with only a global (irrelevant) phase, Arg $\{1/[(\kappa/2 - i\omega)^2 - |\alpha|^2]\}$. Thus we get the final result

$$\hat{b}_{\rm out}[\omega] = \sqrt{G(\omega)}\hat{b}_{\rm in}[\omega] + \sqrt{G(\omega) - 1}\left(\hat{b}_{\rm in}[-\omega]\right)^{\dagger} . \tag{87}$$

Note now that the gain is dependent on ω and on the cavity linewidth (cavity dissipation). At $\omega = 0$ the system becomes unstable for too strong pumping $|\alpha| \geq \kappa/2$.

This type of amplifier, which uses only one mode, is called a **degenerate parametric amplifier**. In this case, at parametric resonance, a pump photon with energy $\hbar\omega_{\rm p}$ splits into two cavity photons, each with energy $\omega_c = \omega_{\rm p}/2$. The simplest generalization of this amplifier is the **nondegenerate parametric amplifier**, where one photon from the pump is split into two photons with different energies (usually corresponding to two modes propagating into different waveguides), such that the sum of the energies of these two photons equals $\omega_{\rm p}$.

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