PHYS-C0252 - Quantum Mechanics Part 2
Section 4.3

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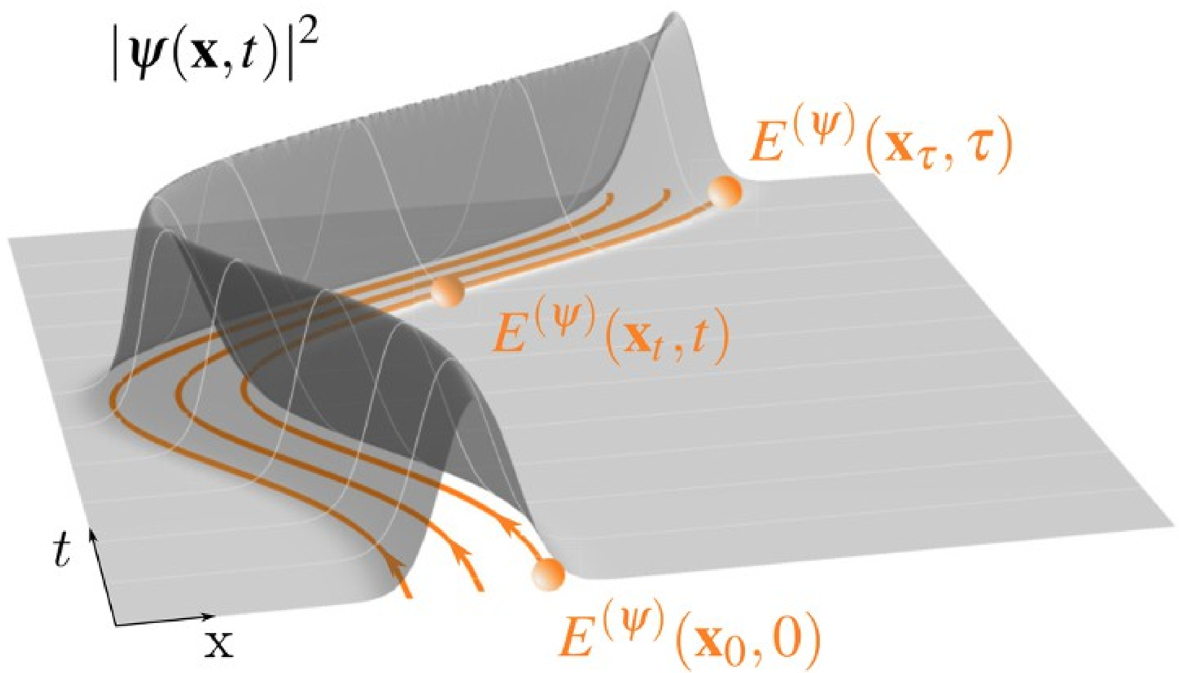
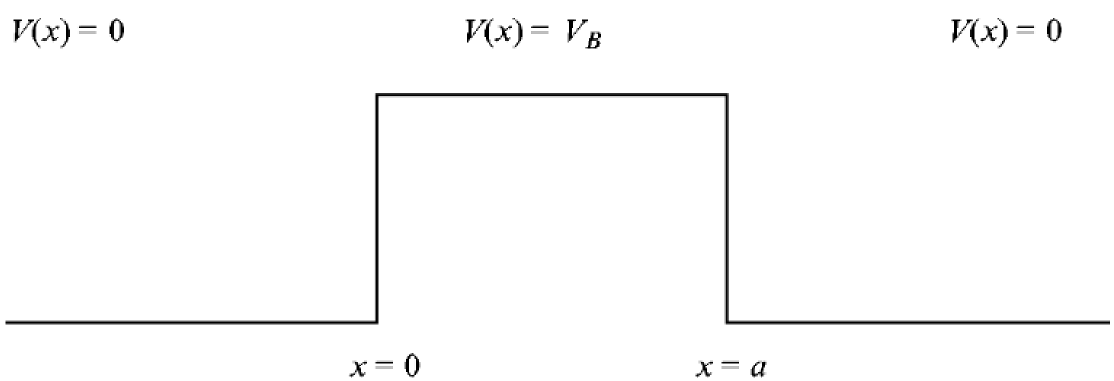


4.3 Finite Potential Barrier

- The first nontrivial case is that of a finite potential barrier, where the QM particle can penetrate in and scatter from:



$$V(x) = \begin{cases} 0 & \text{if } -\infty < x < 0 \\ V_B & \text{if } 0 < x < a \\ 0 & \text{if } a < x < +\infty. \end{cases}$$

The full solution of this problem would entail dynamical treatment, and according to the SE

$$\Psi(x,t) = \int c(E)\psi_E(x)e^{-\imath Et/\hbar}$$

To the left of the barrier

$$\psi''_E(x) = -k^2\psi_E(x), \;\; E = \frac{\hbar^2 k^2}{2m}$$

and the wave solution is

$$\psi_E(x) = A_I e^{\imath kx} + A_R e^{-\imath kx}$$

where the intensity of the *incident* wave is $|A_I|^2$ and that of the *reflected* wave $|A_R|^2$

- When the energy of the incoming particle is larger than that of the barrier (classical crossing)

$$\psi''_E(x) = -k_B^2\psi_E(x), \;\; E = \frac{\hbar^2 k_B^2}{2m} + V_B$$

whose general solution is

$$\psi_E(x) = Ae^{\imath k_B x} + A'e^{-\imath k_B x}$$

For *E* < *V*, the region is classically forbidden (reflection), but the SE gives

$$\psi_E''(x) = \beta^2 \psi_E(x), \; E = -\frac{\hbar^2 \beta^2}{2m} + V_B$$

and the general solution becomes a decaying one

$$\psi_E(x) = Be^{-\beta x} + B'e^{\beta x}$$

Finally, on the r.h.s. of the barrier (equals l.h.s.)

$$\psi_E(x) = A_T e^{\imath k x}, \; k = \sqrt{2Em}/\hbar$$

The physically interesting quantities here are the ratios of the reflected and transmitted intensities

$$R = \frac{|A_R|^2}{|A_I|^2} \quad \text{and} \quad T = \frac{|A_T|^2}{|A_I|^2}.$$

These are called *reflection and transmission probabilities* and $R + T = 1$

We focus here on a particle whose energy is below the barrier:

$$\psi_E(x) = \begin{cases} A_I\,\mathrm{e}^{+ikx} + A_R\,\mathrm{e}^{-ikx} & \text{if } -\infty < x < 0 \\ B\,\mathrm{e}^{-\beta x} + B'\,\mathrm{e}^{+\beta x} & \text{if } 0 < x < a \\ A_T\,\mathrm{e}^{+ikx} & \text{if } a < x < \infty, \end{cases}$$

Continuity at $x = 0$ and a gives

$$A_I + A_R = B + B' \quad \text{and} \quad ikA_I - ikA_R = -\beta B + \beta B',$$

$$B\,\mathrm{e}^{-\beta a} + B'\,\mathrm{e}^{+\beta a} = A_T\,\mathrm{e}^{ika} \quad \text{and} \quad -\beta B\,\mathrm{e}^{-\beta a} + \beta B'\,\mathrm{e}^{+\beta a} = ikA_T\,\mathrm{e}^{ika}$$

from which we can get the amplitudes as a function of *B:*

$$2ikA_I = -(\beta - ik)B + (\beta + ik)B'$$

$$A_T\,\mathrm{e}^{ika} = \frac{2\beta}{(\beta - ik)}B\,\mathrm{e}^{-\beta a} \quad \text{and} \quad B' = B\,\mathrm{e}^{-2\beta a}\frac{(\beta + ik)}{(\beta - ik)}$$

In the limit of a wide barrier where $e^{-2\beta a} \ll 1$ we can approximate that *B'* << *B*, i.e. $2ikA_I \approx -(\beta - ik)B$ which gives

$$A_T\,\mathrm{e}^{ika} \approx -\frac{4ik\beta\,\mathrm{e}^{-\beta a}}{(\beta - ik)^2}A_I$$

and

$$T \approx \left[\frac{16k^2\beta^2}{(\beta^2 + k^2)^2}\right]\mathrm{e}^{-2\beta a}$$

Using the definitions

$$k = \frac{\sqrt{2mE}}{\hbar} \quad \text{and} \quad \beta = \frac{\sqrt{2m(V_B - E)}}{\hbar},$$

this can be written as

$$T \approx \left[\frac{16E(V_B - E)}{V_B^2}\right] e^{-2\beta a}$$

- This is also known as the (QM) *tunneling probability*