



Aalto University
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CS-E5745 Mathematical Methods for Network Science

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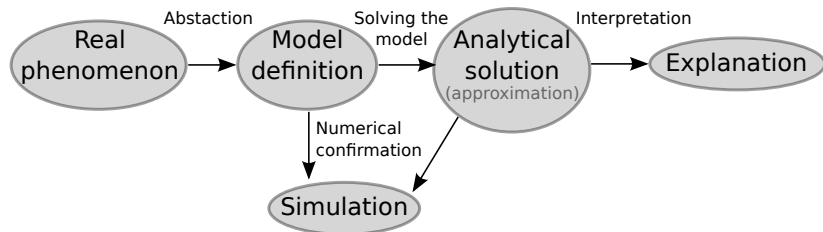
Dynamical models of/on networks

- ▶ Learning goals this week:
 - ▶ Gain insights on the big picture on how to approach network dynamic problems analytically
 - ▶ Ability to recognize and formulate approximations for these problems
 - ▶ Ability to write down master/rate equations
- ▶ Newman: *Networks, An Introduction* - Sections 14 and 17

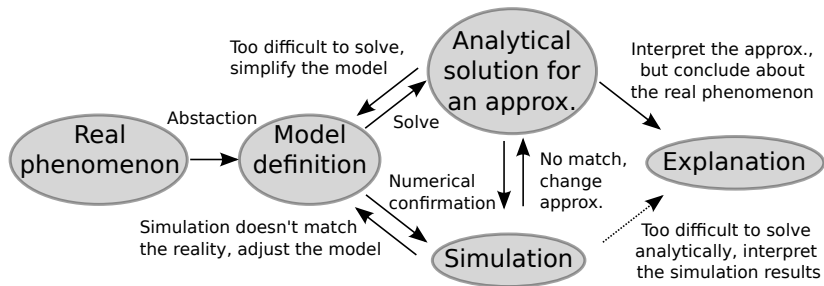
Dynamical models of/on networks

- ▶ Models of networks: Network evolves according to some rules
- ▶ Models on networks: States of nodes evolve according to some rules
- ▶ Co-evolution models: Both network and states of nodes evolve simultaneously
- ▶ Similar techniques can be used to solve all of these

Pipeline, as presented in a publication



Pipeline, a more realistic picture



Example: Barabási-Albert model

- ▶ Phenomena: Networks are born via growth, new nodes are more likely to connect to already popular nodes
- ▶ Data: Probability of link creation can be measured as a function of degree, power-law degree distributions
- ▶ Model: A variation of preferential attachment models (e.g., Price model):
 0. Start with t_0 seed nodes with some connections, and set time to $t = t_0$
 1. Increase time $t = t + 1$, add node with index t
 2. Add m links from t to other nodes, each link to node i prob.
$$p_i = \frac{k_i}{\sum_j k_j}$$
 3. GOTO 1.
- ▶ There could be multi-links (or the probabilities are different), but we ignore this

Example: Barabási-Albert model

- ▶ We can solve the BA model *exactly* if we so wish
- ▶ Idea: Think about the BA model as a Markov chain where the state space is (a subset of) all graphs
 - ▶ Easy to write the transition probabilities between graphs
 - ▶ The result give us a probability distribution over all graphs for each time t

Example: Barabási-Albert model

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- ▶ Idea: Think about the BA model as a Markov chain where the state space is (a subset of) all graphs
 - ▶ Easy to write the transition probabilities between graphs
 - ▶ The result give us a probability distribution over all graphs for each time t
- ▶ Problem: the size of the state space explodes
 - ▶ $2^{T(T-1)/2}$ graphs with T nodes, but not all reachable
 - ▶ For every state at time t , there are $\binom{t}{m}$ possible transitions
 - ▶ Total number of states at time T is $N_T = \prod_{t=t_0}^T \binom{t}{m} \geq e^{mT}$ (for T large enough)
- ▶ This solution is not practical

Approximate solutions

- ▶ Idea: Define a simple system that mimics the (expected) behavior of a complex system
- ▶ The state of the system at each time is reduced to a property or a vector of properties
 - ▶ The choice of properties is usually guided by the process
- ▶ The evolution rules are written in terms of average behavior (master/rate equation)
 - ▶ Either as difference or differential equation(s)
- ▶ Only mean behavior matters: statistical fluctuations are ignored
 - ▶ The properties are assumed to be independent
 - ▶ Dependencies can be added by including combinations of properties
- ▶ The resulting system is usually much easier to solve!

Typical procedure in approximate solutions

- ▶ The “real” model for the system S is defined using stochastic rules F such that $S(t) = F(S(t-1))$
- ▶ Calculate some properties X of the system $X(S) = \mathbf{X}$
 - ▶ The properties can be discrete in the system S but evolve to be continuous in \mathbf{X} : e.g., probability that an edge exists
- ▶ Define deterministic rules f for the evolution of the properties $\mathbf{X}(t) = f(\mathbf{X}(t-1))$, s.t. $f(\mathbf{X}) \approx \langle F(S) | X(S) = \mathbf{X} \rangle$
 - ▶ Usually defined in elementwise way: $X_i(t) = f_i(\mathbf{X}(t-1))$
- ▶ Set $X(S(t_0)) = \mathbf{X}(t_0)$ and solve $\mathbf{X}(t) = f(\mathbf{X}(t-1))$ for any value of t or for $t \rightarrow \infty$
 - ▶ We now hope that $X(\langle F^t(S(t_0)) \rangle) \approx f^t(\mathbf{X}(t_0))$
 - ▶ For continuous time one defines $\frac{d}{dt} \mathbf{X}(t) = f(\mathbf{X}(t))$

Example: Barabási-Albert model

- ▶ Idea: map each graph to a vector where each element corresponds to an edge, s.t. $X_i = 1$ if edge exists, and 0 otherwise
 - ▶ Expected value of binary variable is the probability of getting 1
 - ▶ Notation: A_{ij} is the probability that link between i and j exists
 - ▶ Set $m = 1$ for simplicity
- ▶ The update rule for $\mathbf{X}(t)$:
 - ▶ Calculate expected degree for each node $\langle k_i \rangle = \sum_j A_{ij}$
 - ▶ Set $A_{it} = \frac{\langle k_i \rangle}{\sum_j \langle k_j \rangle}$
- ▶ The final state $\mathbf{X}(T)$ gives the probability of each edge existing at time T

Practical guidelines for selecting variables for approximations

- ▶ Anything that can be measured from the state of the system could be selected as a property we track
 - ▶ Some selections make it difficult to come up with the update function
 - ▶ The selecting variables appearing in the update rule is a good idea
- ▶ The above approximation discards dependencies between the variables, but one can define a combination variables that keep track of these
 - ▶ Example: Probability for each triangle to exists, instead of a probability each link to exists

Example: Barabasi-Alberts model

- ▶ Idea 1: map each graph into a vector of degrees of nodes
 - ▶ The expected degree of each node is tracked
 - ▶ Note: expected degrees are continuous variables
- ▶ $X_i(t)$ is the expected degree of node i at time step t
- ▶ Now we should write the equations $X_i(t) = f_i(\mathbf{X}(t - 1))$

Example: Barabasi-Alberts model

- ▶ Idea 2: map each graph into a vector of counts of nodes of each degree (or a degree distribution)
 - ▶ The expected number of nodes of each degree is tracked
- ▶ $X_k(t)$ is the expected number of nodes of degree k at time step t
- ▶ Now we should write the equations $X_k(t) = f_k(\mathbf{X}(t - 1))$

Practical guidelines for writing master equations

- ▶ When the tracked variables are not binary variables, it might be more difficult to write the update equations
- ▶ Think about each element $X_i(t)$ separately, divide it to
 - ▶ *growth* $X_i^+(t) = f_i^+(\mathbf{X}(t-1))$ term and
 - ▶ *decline* $X_i^-(t) = f_i^-(\mathbf{X}(t-1))$ term
- ▶ The total is
$$X_i(t) - X_i(t-1) = X_i^+(t) - X_i^-(t) = f_i^+(\mathbf{X}(t-1)) - f_i^-(\mathbf{X}(t-1))$$
- ▶ Are there any conserved quantities? (If something grows, others decline)
- ▶ Same applies for rate equations

Example: Barabasi-Alberts model

- ▶ $X_k(t)$ is the expected number of nodes of degree k at time step t
- ▶ Addition of nodes to degree categories:
 - ▶ $X_k^+(t) = m \frac{(k-1)}{\sum_k kX_k(t-1)} X_{k-1}(t-1)$, when $k > m$
 - ▶ $X_k^+(t) = 1$, when $k = m$
- ▶ Removal of nodes from degree categories:
 - ▶ $X_k^-(t) = m \frac{k}{\sum_k kX_k(t-1)} X_k(t-1)$
- ▶ Note that $\sum_k kX_k(t) = 2mN(t)$

Example: Barabasi-Alberts model

- ▶ In total we have:

- ▶ $X_k(t) = X_k(t-1) + \frac{(k-1)}{2N(t-1)}X_{k-1}(t-1) - \frac{k}{N(t-1)}X_k(t-1)$,
when $k > m$

- ▶ $X_m(t) = X_m(t-1) + 1 - \frac{m}{N(t-1)}X_m(t-1)$

- ▶ At the stationary state we should have:

$$x_k = X_k(t)/N(t) = X_k(t-1)/N(t-1):$$

- ▶ $x_k = \frac{k-1}{k+2}x_{k-1}$, when $k > m$

- ▶ $x_m = \frac{1}{1+m/2}$

Models on networks

- ▶ Above approach can also be used to solve models *on* networks
- ▶ The state of the systems is not a graph, a vector of states for the nodes
- ▶ Examples:
 - ▶ Infection spreading: Nodes are either infected or not infected
 - ▶ Opinion formation: Each nodes state represent its opinion (discreet or continuous value)

Spreading models on networks

- ▶ Network spreading models follow the classic literature on mathematical epidemiology
- ▶ Models are determined by the possible states and transition rates between them
- ▶ States:
 - ▶ S: Susceptible
 - ▶ E: Exposed
 - ▶ I: Infected
 - ▶ R: Recovered (or Removed)
- ▶ Naming convention: state changes from left to right
- ▶ Typical models include: SI, SIS, SIRS, SEIR

Example: SIR on networks

- ▶ $S \xrightarrow{\beta} I \xrightarrow{\gamma} R$
- ▶ Classic result (no network): Epidemic if $R_0 = \frac{\beta}{\gamma} > 1$
 - ▶ i.e., when rate of infection is larger than rate of recovery
- ▶ Degree-based approximation: Epidemic if $\frac{\beta}{\gamma} > \frac{1}{\langle q \rangle}$
 - ▶ where $\langle q \rangle$ is the expected excess degree
- ▶ Node-based approximation: Epidemic if $\frac{\beta}{\gamma} > \frac{1}{\lambda_1}$
 - ▶ where λ_1 is the largest eigenvalue of an adjacency matrix

SIR can be mapped to percolation

- ▶ Modification: all individuals stay infected for time τ (instead of $I \xrightarrow{\gamma} R$)
- ▶ Infected node: probability that an *edge* leading out is activated during infection is $\Phi = 1 - e^{-\beta\tau}$
- ▶ Set edges occupied with probability $\Phi \rightarrow$ components give you the possible sizes of epidemics
 - ▶ We can now calculate the statistics of epidemics using percolation theory on networks

Binary state models on networks

- ▶ A general theory for approximating processes where each node can be in one of two states¹
 - ▶ Examples: SI, SIS, voter model, ...
 - ▶ Assumption: network is produced by configuration model
- ▶ Probabilities that node of degree k is susceptible (infected) and has m infected neighbors: $s_{k,m}$ ($i_{k,m}$)
- ▶ Dynamics determine the rates $S \rightarrow I$: $F_{k,m}$ and $I \rightarrow S$: $R_{k,m}$
 - ▶ Example: in the SIS model $F_{k,m} = m\beta$ and $R_{k,m} = \gamma$

¹Phys. Rev. Lett. 107, 068701 (2011)

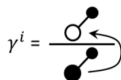
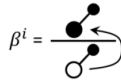
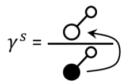
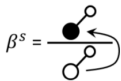
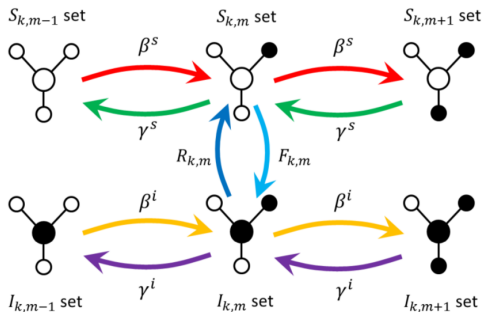
Binary state models on networks

- ▶ The master equations become complicated:

$$\begin{aligned}\frac{ds_{k,m}}{dt} = & -F_{k,m}s_{k,m} + R_{k,m}i_{k,m} \\ & -\beta^S(k-m)s_{k,m} + \beta^S(k-m+1)s_{k,m-1} \\ & -\gamma^S m s_{k,m} + \gamma^S(m+1)s_{k,m+1}\end{aligned}$$

- ▶ Where $\beta^S = \frac{\langle \sum_{m=0}^k (k-m) F_{k,m} s_{k,m} \rangle}{\langle \sum_{m=0}^k (k-m) s_{k,m} \rangle}$, $\gamma^S = \frac{\langle \sum_{m=0}^k (k-m) R_{k,m} i_{k,m} \rangle}{\langle \sum_{m=0}^k (k-m) i_{k,m} \rangle}$
- ▶ Similar equation for $i_{k,m}$

Binary state models on networks



Binary state models on networks

- ▶ The $(k_{\max} + 1)(k_{\max} + 2)$ master equations can be solved numerically, or further approximations can be made
- ▶ Pair approximation (PA):
 - ▶ ρ_k : Probability that node of degree k is infected
 - ▶ ρ_k : Probability that neighbor of a node of degree k is infected
 - ▶ $s_{k,m} \approx (1 - \rho_k)B_{k,m}(\rho_k)$, where $B_{k,m}(\rho) = \binom{k}{m}\rho^m(1 - \rho)^{k-m}$
- ▶ Mean-field approximation (MF):
 - ▶ ρ_k : Probability that node of degree k is infected
 - ▶ $s_{k,m} \approx (1 - \rho_k)B_{k,m}(\langle \frac{k}{Z}\rho_k \rangle)$

Binary state models on networks

