Lecture 3 Wave mechanics

1. Wave formation

What are water waves? How do they form and how do we describe them? Looking out at the ocean, one often sees a seemingly infinite series of waves, transporting water from one place to the next. Though waves do cause the surface water to move, the idea that waves are travelling bodies of water is misleading. In naval architecture waves generated by local winds are termed **sea** and waves that travelled out of their area of generation are termed **swell**. Sea waves are characterized by relatively peaky crests and the crest length seldom exceeds some two or three times the wavelength. Swell waves are generally lower with more rounded tops. The crest length is typically six or seven times the wavelength. In a swell, the variation in height between successive waves is less than it is for the case for sea waves.

Waves are created by anything that supplies energy to the water surface. Consequently, sources of wave systems are numerous. From our own experience we know that throwing a stone into a pool will generate a circular wave pattern. When examining the wave system creation sequence, it is important to be aware of the energy transfer that is constantly occurring in a wave. The energy of a wave is always dissipated by the viscous friction forces associated with the viscosity of the sea. This energy dissipation increases with wave height.

For a wave to be maintained, the energy being dissipated must be replaced by the energy source, the wind. Hence, without the continued presence of the wind, the wave system will die. Deep water ocean waves are essentially energy passing through the water causing it to move in a circular motion. When a wave encounters a surface object, the object appears to lurch forward and upward with the wave- Then it falls back in an orbital rotation as the wave continues to progress, ending up in the same position as before the wave appeared. If one imagines wave water itself following this same pattern, it is easier to understand ocean waves as simply the outward manifestation of kinetic energy propagating through seawater. On this basis it is fair to accept that the passing of air over the sea surface causes ripples or waves.

In 1957 two key theories based on potential flow assumptions on the generation of waves were introduced. The theory of Philips (1957) suggested that the generation of waves upon a water surface that is originally at rest is the result of a random distribution of normal pressure associated with the onset of a turbulent wind. Philips believed that waves develop most rapidly by means of a resonance mechanism which occurs when a component of the surface pressure distribution moves at the same speed as the free surface wave with the same wave number. The second theory by Miles (1957) suggested that the generation of surface waves is the result of a shear flow. Thus, the rate at which energy is transferred to a wave of certain speed is proportional to the profile curvature at that elevation. Notwithstanding these developments and various other research studies, until today the concept of the wave formation mechanism by which the energy is transferred from wind to sea water is not fully understood. What we believe that both theories by Philips and Miles are reallisitc and therefore while waves are enlarged by shear forces, they interact and combine to form longer waves and therefore fully developed seas. The wave disturbance depends on the wind strength, the time for which wind acts on the sea surface, and the portion of the water surface it acts upon it. Those features are respectively termed **wind strength, duration and fetch**.

The idea of waves being an energy transmission medium (rather than water movement) makes sense in open seas. In coastal areas, where waves are clearly seen crashing dramatically onto shore this phenomenon is a result of the wave's orbital motion being disturbed by the seafloor. As a wave passes through water, not only does the surface water follow an orbital motion, but a column of water below it (down to half of the wave's wavelength) completes the same movement. The approach of the bottom in shallow areas causes the lower portion of the wave to slow down and compress, forcing the wave's crest higher in the air. Eventually this energy imbalance reaches a breaking point. The crest comes crashing down and wave energy is dissipated into the surface.

(a) Ocean waves

(b)The wave energy cycle

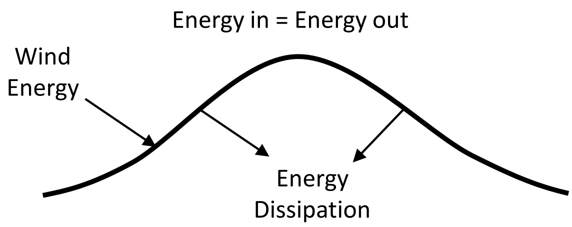


Figure 3-1 Principles of wave formation

Where does a wave's energy come from? There are a few types of ocean waves and they are generally classified by the energy source that creates them. Most common are surface waves, caused by **wind** blowing along the air-water interface, creating a disturbance that steadily builds as wind continues to blow and the wave crest rises. Surface waves occur constantly all over the globe and are the waves you see at the beach under normal conditions (Sundar, 2015). **Adverse weather or natural events** often produce larger and potentially hazardous waves. Severe storms moving inland often create a storm surge, a long wave caused by high winds and a continued low-pressure area. **Submarine earthquakes or landslides** can displace a large amount of water very quickly, creating a series of very long waves called tsunamis. Storm surges and tsunamis do not create a typical crashing wave but rather a massive rise in sea level upon reaching shore. They can be extremely destructive to coastal environments. Once a wave has been generated it will move away from the position at which it was generated until all its energy is spent (Ochi, 1993; Cherneva and Guedes Soares, 2014).

2. Brief overview of wave theories

Extensive explanation to the background of wave theories is given by Moctar et al. (2021) and Karadeniz et al. (2013). This section outlines their basic taxonomy and basic potential flow theory assumptions. Airy waves (known as linear water surface waves) are defined on the basis of potential flow theory assumptions. Accordingly, the fluid flow is assumed to be inviscid, incompressible and irrotational. On the basis of continuity, the mass and momentum are defined respectively by Laplace and Bernoulli equations as follows

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \tag{3-1}$$

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}(\nabla\phi)^2+\frac{p}{\rho}+gz=0 \tag{3-2}$$

where ϕ represents the velocity potential; ρ is the density of water, p the pressure; g is the acceleration of gravity, t the time and z the water surface elevation. The boundary conditions for the above equations are defined as kinematic and dynamic. Kinematic conditions relate to the motions of the water particles. **Dynamic boundary conditions** relate with the forces acting on water particles. The dynamic boundary condition at the sea surface assumes that the pressure is always the atmospheric pressure (taken as the reference pressure = 0), and the wave is only subject to gravity forces. A linear wave is assumed to be periodic and infinitely long in the longitudinal direction. Variations in the transverse direction are ignored and the system boundaries limit in way of the water surface and the seabed (see Figure 3-2). The **kinematic boundary condition** at the water surface is based on the assumption that the particles may not leave the water surface. This means that the velocity of the water particle normal to the surface (u_z) is equal to the speed of the surface in that direction. At the sea surface (z=0) we assume that:

$$u_z=\frac{\partial\zeta}{dt}\rightarrow\frac{\partial\phi}{\partial z}=\frac{\partial\zeta}{dt} \tag{3-3}$$

At the bottom (z=-d), the water particles may not penetrate the seabed and accordingly we can assume that:

$$u_z=0\rightarrow\frac{\partial\phi}{\partial z}=0 \tag{3-4}$$

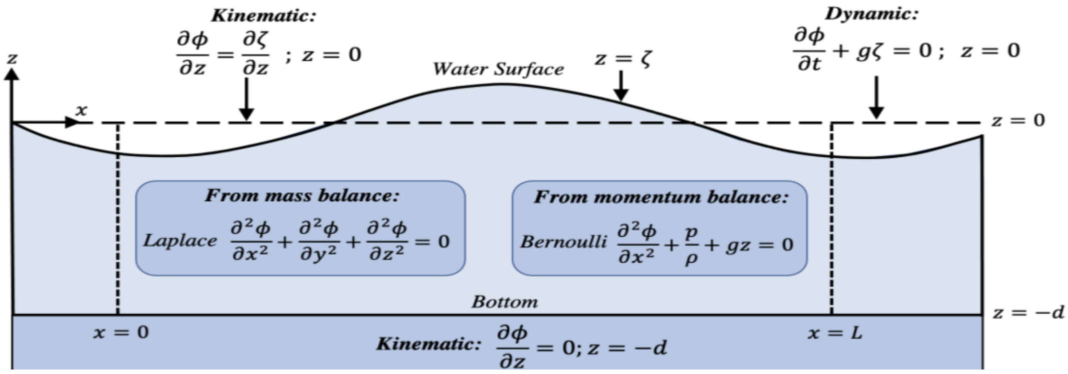


Figure 3-2 Linear wave theory idealization

Generally, flow disturbances propagate in various directions interacting non-linearly over water of probable non-uniform density and varying or deformable topography. A well-known nonlinear theory is **Stokes theory**. It is considered most suitable for waves which are not very long relative to the water depth. It assumes that all the variations in the longitudinal direction can be represented by Fourier series and that the coefficients in these series can be written by a perturbation expansion defined through parameters that increase as the wave height increases. By substituting the higher order perturbation expansions in the governing equations (i.e., the mass and the momentum balance equation) yields to solving the velocity potential. Stokes wave theories are most suitable for deep and intermediate water depth.

For shallow waters, a finite-amplitude wave theory is required. **Cnoidal wave theory** and, in very shallow waters, **solitary wave theory** are the analytical wave theories most commonly used. Solutions in the cnoidal wave theory are obtained by the use of elliptical integrals of the first kind. The solitary wave theory is a special case combining mathematical principles of cnoidal and linear wave theories. As the relative depth decreases the cnoidal wave becomes the solitary wave, which has a crest that is completely above the still water level and has no trough.

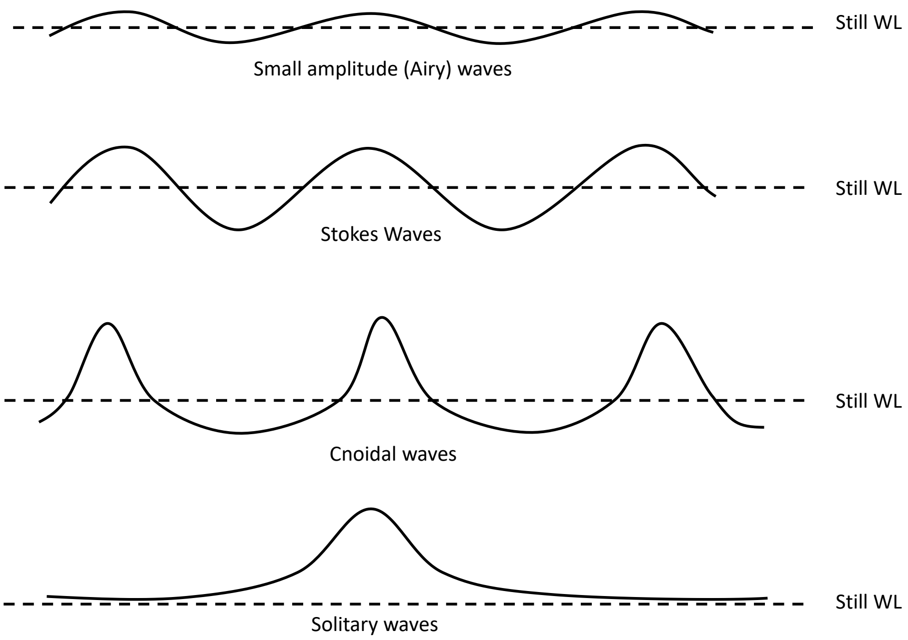


Figure 3-3 Different forms of wave surface elevation (SWL : idealization of Still Water Line)

3. The regular wave

Evaluation of the properties of random waves is almost impossible. However, if we consider the randomly changing waves as a stochastic process, then it is possible to evaluate the statistical properties of irregular waves as an aggregate of regular waves. Regular waves are shaped like a sinusoidal wave moving along water surface. This type of waves is periodic, meaning it has consistent frequency or period of occurrence. Figure 3-4 gives the classic notation used to describe the characteristics of a regular wave. The wave is progressive, meaning it moves horizontally over the water surface.

If we were to consider only a single point in space and describe the water surface elevation at that point as the waves move past, then the wave elevation in time is defined as

$$\zeta(t) = \zeta_0 sin(\omega t\text{-}\varepsilon) \tag{3-5}$$

In the above expression t is the variable for time, ζ_0 is the wave amplitude, ε is the phase angle (the degrees the shape is different from a perfect sine wave) and ω is the wave frequency (in radians / second) presenting a measure of the oscillations that pass this point in one second. If we instead consider the entire wave train in space, but only for a single moment in time, the mathematical expression becomes

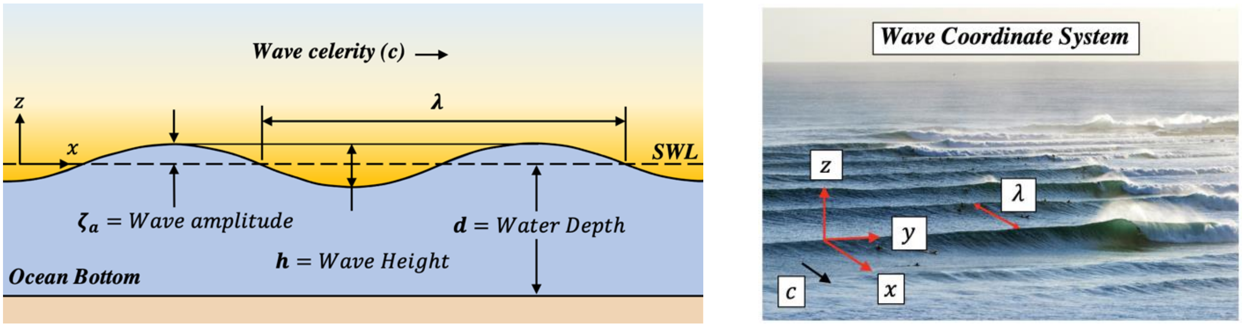
$$\zeta(x) = \zeta_0 sin(kx) \tag{3-6}$$

where x is the variable for position and k is the wave number representing the frequency as a function of wavelength - λ (i.e. the number of cycles that occur over a unit of length). The expression for wave number is as follows

$$k = \frac{2\pi}{\lambda} \tag{3-7}$$

(a) 2D surface elevation

(b) 3D surface formation



ζ : instantaneous depression of water surface below mean level (y = 0)
ζ_a : wave amplitude from mean level ($y = 0$) to a crest or trough
H : wave height (always twice the wave amplitude)
λ : wave length (distance from one crest – or trough – to the next)
c : wave celerity
T : wave period (time interval between successive crests or troughs passing a fixed point
α : the instantaneous wave slope (gradient of the surface profile)
α_0 : maximum wave slope or wave slope amplitudes
H/λ : wave steepness

Figure 3-4 Regular wave idealization

Waves exist and change in both time and space. So, you may stay with a wave and move through space in time or you can stay at one location and see the wave move past in time. Accordingly, the equation for the water elevation must account for the point at which we are measuring (i.e., where are we standing?) and the time the measurement is made (i.e., what time are we looking?). On this basis Equation (3-5) can be expressed as

$$\zeta(t) = \zeta_0 sin(kx - \omega t\text{-}\varepsilon) \tag{3-8}$$

For regular deep water waves there is a fixed relationship between wave frequency, length and speed. For a high frequency wave there is only a short time between peaks and therefore the wave length is very short. On the other hand, for low frequency waves there is a long time between peaks and the wave length is long. For deep waters the key relationships describing wave period and length are as follows

$$T = \frac{2\pi}{\omega} \tag{3-9}$$

$$\lambda = \frac{2\pi g}{\omega^2} = \frac{gT^2}{2\pi} \tag{3-10}$$

In shallow waters the wavelength (λ) depends only on the water depth (*d*). Thus

$$\lambda = 2\pi d \tag{3-11}$$

The general relationship between wave frequency, length and depth is given by the so called *"dispersion equation"*

$$\omega^2 = gk \times tanh(kd) = gk\frac{sinh(kd)}{cosh(kd)} \tag{3-12}$$

where $kd = \frac{2\pi \mathrm{d}}{\lambda}$ and therefore when the water depth is very large relative to the wavelength *kd* is large. Vice versa in shallow waters when depth is small relative to the *kd* becomes small. For these situations we the terms in the hyperbolic expressions of Equation (3-12) take the form described in Table 3-1.

Table 3-1 Forms of hyperbolic expression

Function	**Shallow waters**	**Deep waters**
Sinh(kd)	$e^{kd}/2$	1
Cosh(kd)	$e^{kd}/2$	*kd*
Tanh(kd)	1	*kd*

Using these simplifications of the hyperbolic functions we can show that in deep water the wave frequency depends only on the wavelength,

$$\omega = \sqrt{gk} \tag{3-13}$$

while in shallow water the wave frequency also depends on water depth

$$\omega = \sqrt{gk^2d} \tag{3-14}$$

The **wave celerity *(c)***, i.e. the speed of the wave travelling over the water surface is given by:

$$c = \sqrt{\frac{g}{k}\tanh{(kd)}} \tag{3-15}$$

As with wavelength, the equation takes different forms in deep and shallow water conditions namely

$$c = \sqrt{\frac{g}{k}} = \frac{g}{\omega} \tag{3-16}$$

$$c = \sqrt{gd} \tag{3-17}$$

The above relationships helps explain why although tsunamis are difficult to observe out in the ocean, they may develop into towering waves as they approach the shore. In deep waters the tsunami wave has a very long wavelength and is travelling extremely fast. However, as the wave approaches the shore the wave speed and length are determined by the water depth. In this process and while the wave slows down the kinetic energy is transformed into energy stored in wave amplitude (i.e., potential energy).

While wave celerity can give the velocity of the crest of a wave moving over the water surface the wave group velocity gives the velocity of energy associated with the wave. The concept can be demonstrated in a wave tank. When the wave maker sends the first regular wave train down the tank you may witness the height of the wave decreasing as it travels. Eventually that first wave disappears. In deep water conditions the energy in the wave (which is seen in the wave height) is travelling as fast as the wave crest.

The **total energy** associated with a train of regular waves includes contributions from both potential and kinetic energy (see Figure 3-5)**.** If we assume a portion of regular wave idealisation with height ζ, the center of gravity of this is located in the middle i.e., in way of ζ /2 and has a mass $\rho g \delta x$. The potential energy (*P.E.*) of this portion is defined as

$$P.E. = mgh = (\rho g \delta x)\frac{\zeta}{2} = \frac{\rho g \zeta^2 \delta x}{2} \tag{3-18}$$

If we integrate this energy over the entire wavelength, we get the potential energy (E_{PE}) of the wave per unit width as

$$E_{PE} = \frac{\rho g \lambda \zeta_0^2}{4} \tag{3-19}$$

If the total velocity of the segment is q then the kinetic energy of the segment is

$$K.E. = \frac{1}{2}mq^2 = \frac{\rho q \delta x \delta z}{2} \tag{3-20}$$

Consequently, integrating this over the full wavelength and by utilizing the relationship between the speed and wavelength gives the total *K.E.* (E_{KE})per unit width as

$$E_{KE} = \frac{\rho g \lambda \zeta_0^2}{4} \tag{3-21}$$

Thus the total energy becomes

$$E_{TOT} = E_{PE} + E_{KE} = \frac{\rho g \zeta_0^2}{2} \tag{3-22}$$

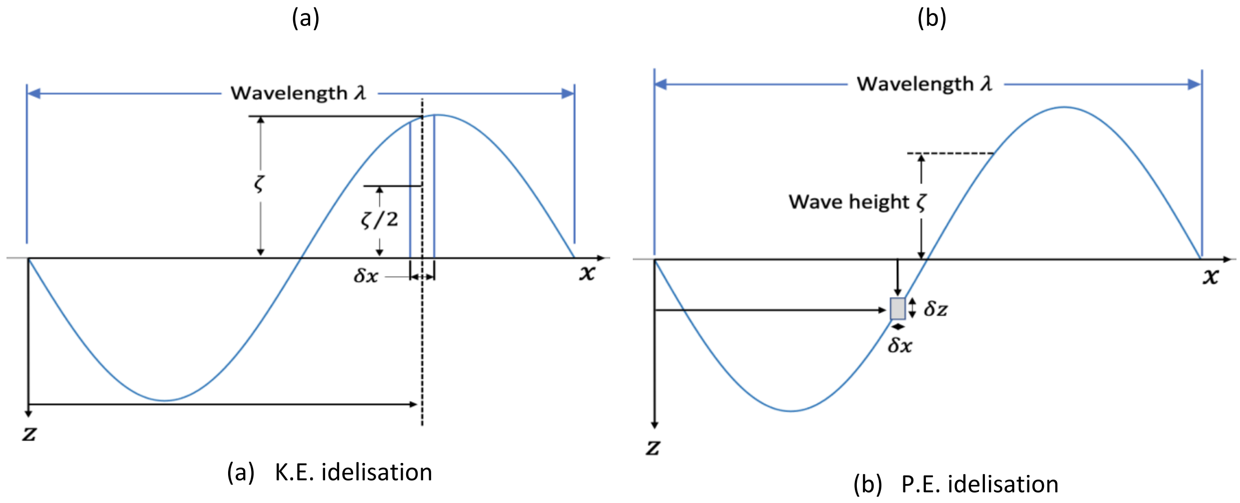


Figure 3-5 Wave energy integration

Deep water idealizations imply that water particles involved in the wave motion do not detect the bottom. For deep water waves, the water particles move in circular motion. This means that the particles are not travelling with the wave, but the wave passes along while the particles stay in the same spot. You have experienced this in the ocean if as waves pass you by. Although the waves move you up and down there is very little sideways motion. The particles near the surface of the water make large circular motions. However, as you go deeper in the water the particle motions decrease in amplitude (see Figure 3-6). Eventually the circular motion becomes so small that the water particles do not move as the wave passes by.

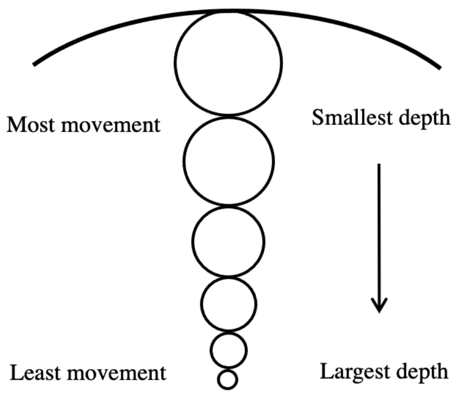


Figure 3-6 Decreasing water particle motion as a function of depth

4. Questions

1. What do we mean by the terms sea and swell? How are they related and what are the key characteristics of each?
2. What are the factors that influence wave formation? How far does water travel in waves?
3. Where does the wave energy come from? What are the main differences between surface waves, storm surges, and tsunamis?

4. When is it appropriate to consider Stokes theory for waves? What are the key differences between regular and Stokes waves?
5. How and where are Cnoidal waves formed? What type of wave category is appropriate to consider for solitary waves?
6. What considerations make it possible to evaluate the statistical properties of irregular waves? When are these assumptions valid?
7. What factors influence the wave frequency in deep and shallow waters?
8. Why is it difficult to detect tsunamis that are far away from shore? How tsunami waves relate to wave celerity?
9. What are the ways in which wave energy is dissipated? Which ones are important to ship maneuvering?
10. How do we describe deep waters and why they are relevant in ship dynamics?

5. References

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