Lecture 4 The statistical representation of irregular waves

1. Introduction

The confused state of the sea at any point can be modeled in 3D as the interference pattern created between several wave systems. These random wave systems form in different phases and at differing distances in relation to an observation point. Modelling irregular seaways is possible by applying the principle of **superposition** according to which the complicated sea wave system is made up of many sinusoidal wave components superimposed upon each other (Whitham, 1999). Each component sine wave has its own wavelength, speed and amplitude and is created from one of the wave energy sources (see Figure 4-1).

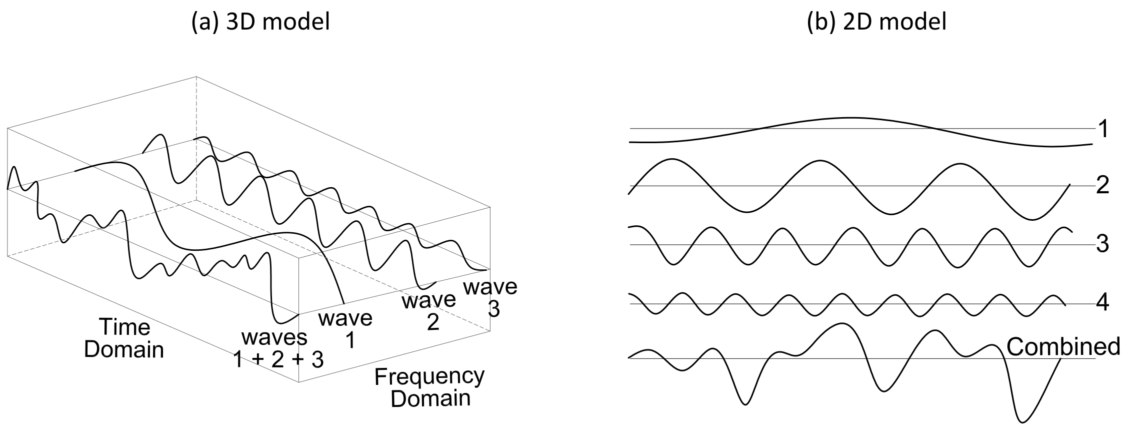
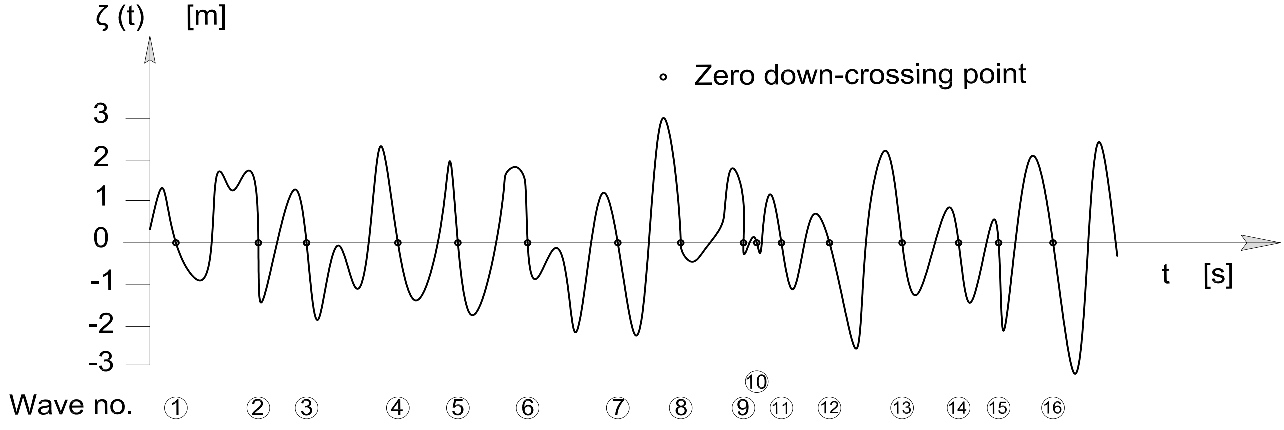


Figure 4-1 The superposition of irregular waves

Irregular sea waves can be understood within the context of Gaussian statistics (see Appendix 1 and Rawson and Tupper, 2001). The amplitude and frequency of the component waves are selected from the wave spectra (mean square spectral density functions) which are a measure of the wave energy. The response to the irregular seaway is therefore a superposition of the responses of each regular wave. This analysis is analogous to the representation of non-sinusoidal periodic excitations and responses using Fourier analysis (see section 20). The main assumptions maintained in the statistical approach is that the sea is stationary, its statistical properties (e.g., average wave and period) do not change within a considered time frame and the seaway is not too steep so that the linear superposition of a regular number of waves and the utilization of the linearized equations are still accurate.

For a regular wave, the wavelength is the distance between two successive crests or troughs. For irregular waves, the individual wave is defined by **two successive zero down-crossings.** In a real wave recording there will be hundreds of individual waves. An example is shown in Figure 4-2. Here an irregular wave trace comprises of 15 individual waves ordered in time by height, zero crossing periods and wave number. A histogram can be used to evaluate the range of water elevation variation (e.g. see Figure 4-3). To create such histogram water surface measurements at a particular location are made at regular intervals (e.g., every 1 minute). The measurements at a particular location are then grouped into elevation ranges.



Rank i	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
$\boldsymbol{H}\ [\boldsymbol{m}]$	5.5	4.8	4.2	3.9	3.8	3.4	2.9	2.8	2.7	2.3	2.2	1.9	1.8	1.1	0.23
$\boldsymbol{T}[\boldsymbol{s}]$	12.5	13	12	11.2	15.2	8.5	11.9	11	9.3	10.1	7.2	5.6	6.3	4	0.9
Wave no.	7	12	15	3	5	4	2	11	6	1	10	8	13	14	9

Figure 4-2 Irregular wave record table measured by a buoy

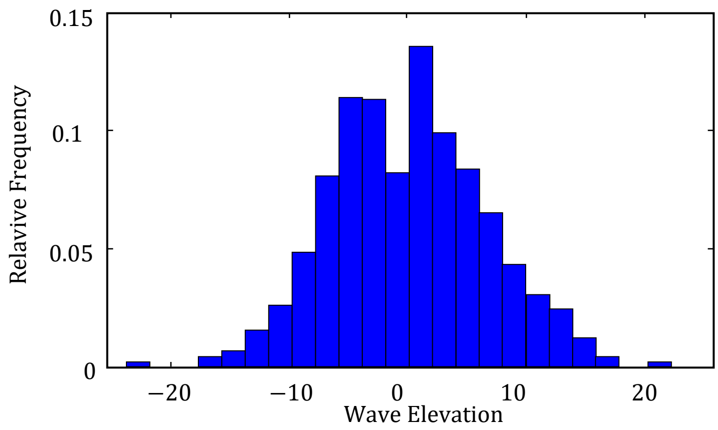


Figure 4-3 Example of water elevation histogram

A more detailed description of sea spectra is given in Michel (1968) and Rawson and Tupper (2001). The easiest statistical measures we can use to quantify wave energy are **means** and **averages**. For example, we can average all the measurements of the water elevation (ζ) to obtain the mean water level $\bar{\zeta}$. We can also measure the mean peak amplitude for every wave in a signal, namely $\bar{\zeta}_a$. Then the mean wave height is defined as twice the mean wave amplitude

$$H_a = 2 \times \bar{\zeta}_a \tag{4-1}$$

The time between each peak can also be measured as T_p and then averaged to give the **mean period** of the peaks $\overline{T_p}$. The time between zero crossings (i.e., the time between the water surface passing up through the nominal zero water level) can be measured for all waves as T_z and averaged as $\overline{T_z}$. Then the mean of any *N* set of numbers (*x*) is given by

$$\bar{x} = \sum_{n=1}^{N} \frac{x_n}{N} \tag{4-2}$$

So, the mean water elevation is:

$$\bar{\zeta} = \sum_{n=1}^{N} \frac{\zeta_n}{N} \tag{4-3}$$

where N is the no. of measurements and ζ_n is each measurement of the water surface.

The variance of a set of numbers is a measure of how spread out the data are (i.e. how far the numbers lie from the mean). Accordingly, the variance of the water elevation (m_0) is written as:

$$m_0 = \sum_{i=1}^{N} \frac{(\zeta_n - \bar{\zeta})^2}{N} \tag{4-4}$$

The standard deviation is another measure of the dispersion of the data. If the standard deviation is small the data points are close to the mean. On the other hand, if the standard deviation is large the data points are spread out over a large range of values. The standard deviation (σ_0) is equal to the square route of the variance (m_0), i.e.

$$\sigma_0 = \sqrt{m_0} \tag{4-5}$$

In general, for results to be meaningful the data must include at least 100 pairs of peaks and troughs.

In ship dynamics a useful concept used to explain waves is the significant wave height (H_s) and significant wave periods (T_s) that are correspondingly defined as the average wave height and period of the one-third of the highest waves. This is because the design of ship structures has been traditionally based on experiences and visual observations of the waves which concentrate only on significant wave heights and periods in specific area of observation (e.g., the North Atlantic). An example of the practical interpretation of statistical distributions is given in the tabulated values presented in Figure 4-2 whereby the 1/3 of the total number of individual waves is 5 and consequently the 5 most significant wave heights are chosen as

$$H_S = H_{1/3} = \frac{1}{5}\sum_{i=1}^{5} H_i = \frac{1}{5}(5.5 + 4.8 + 4.2 + 3.9 + 3.8) = 4.44\ m \tag{4-6}$$

In this case the significant period becomes

$$T_s = T_{1/3} = \frac{1}{5}\sum_{j=1}^{5} T_i = 12.9\ s \tag{4-7}$$

In occasion the significant wave height is taken as the average of only *1/10* of the total number of ensembles ($H_{1/10}$). The maximum wave height H_{max} among a long-time record e.g., 100 years is often used when designing an offshore structure.

In naval architecture practice, the wave height might be denoted by its probability of exceedance. Thus, wave data histograms are usually represented in a non-dimensional form to allow for comparison of waves in different locations. The total wave height is then divided by the average wave height and plotted against the probability density function. The Rayleigh probability density function used in such representations (see Figure 4-4) is defined as:

$$f(H/\bar{H}) = \frac{\pi}{2}(H/\bar{H}) \cdot e^{-\pi/4(H/\bar{H})^2} \tag{4-8}$$

A benefit we can gain from the use of such approach is that we can represent the characteristic wave heights as a function of the average wave height in a histogram. A reservation on the utilization of Rayleigh distribution is that it loses validity when applied to broad spectrum. In such cases the wave height energy spreads over wide range of frequencies.

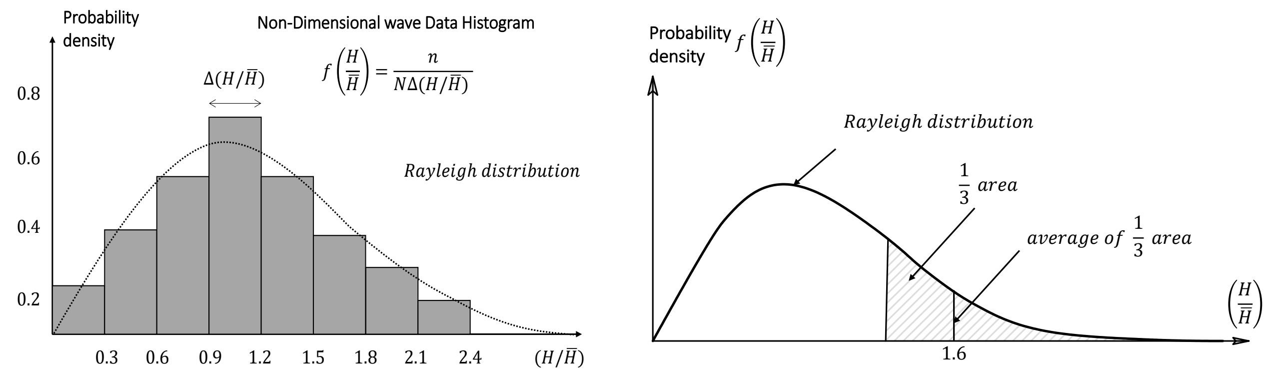


Figure 4-4 Non-dimensional wave data histogram and Rayleigh distribution

2. Wave superposition and Fourier analysis

Let us consider two waves travelling past the same point (say $x = 0$) and let us assume that they have the same amplitude, ζ_0 but different frequencies ω_1, ω_2. Their corresponding wave elevations will be presented as:

$$\zeta_1(t) = \zeta_0 sin(\omega_1 t) \tag{4-9}$$

$$\zeta_2(t) = \zeta_0 sin(\omega_2 t) \tag{4-10}$$

Assuming linear superposition, the water elevation at this point will be the sum of the two waves travelling past, i.e. $\zeta(t) = \zeta_1(t) + \zeta_2(t)$. Figure 4-5 shows how two waves of the same wave amplitude but different frequencies combine to form a new wave. If we change the wave amplitude and add a phase angle the combined wave changes. The more waves of different frequencies, amplitudes and phase angles are used the more complex the resulting water elevation becomes. For example, Figure 4-6 shows a wave train comprising of 50 frequency components. In the same way we can create an irregular wave train by combining many frequency components of different frequencies.

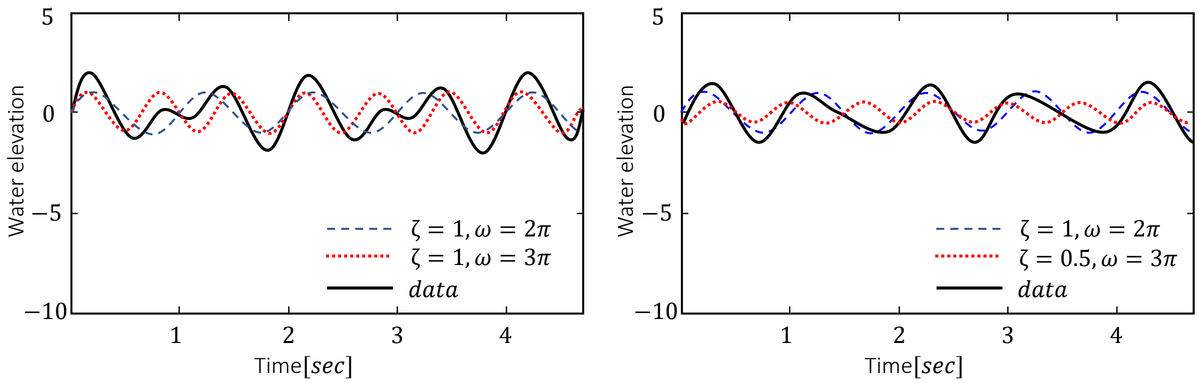


Figure 4-5 Superposition of two waves

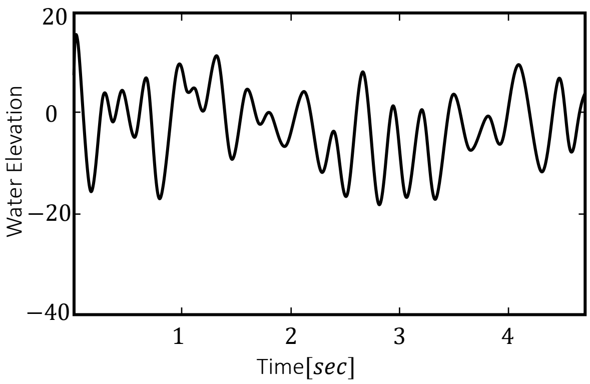


Figure 4-6 Superposition of 50 wave components

The different frequency components in a given wave train can be identified using a **Fourier Transformation (FT)**. An FT helps to identify the amplitude and phase angle of each frequency component contained in a wave. The first step of this process is to consider the wave time history to be represented by the sum of all these components added to the mean water elevation as

$$\zeta(t) = \bar{\zeta} + \sum_{n=1}^{N} \zeta_{n0} sin(\omega_n t + \varepsilon_n) \tag{4-11}$$

where ω_n represents each signal wave frequency and ζ_{n0} and ε_n are the corresponding wave amplitude and phase angle.

In reality all signals contain some noise but in any case, if a wave signal is described by a single sinusoidal function finding the frequency content will result in a single number. The **discrete FT (DFT**) helps to convert a sequence of values corresponding to certain time instants to a sequence of values corresponding to specific frequencies. The most common DFT is the so-called **Fast Fourier Transform (FFT)**. The time domain consists of a set of numbers $(x_0, x_1, \ldots., x_n)$ each measured at a particular time $(t_0, t_1, \ldots., t_n)$. The DFT provides the frequency domain information of numbers $(X_0, X_1, \ldots., X_k)$ where each X_k represents a portion of the signal that occurs at a frequency component, say f_k. By convention, n refers to the time domain data point and k refers to the frequency domain data point. The frequency values of X_k are typically complex numbers expressed in real and imaginary parts. If we have N data points in time (x_n) we will have only $N/2$ points in the frequency domain. This is because for each frequency we have two pieces for information; namely the magnitude and the phase – while at each time we only have one piece of information namely ‘‘*magnitude*’’. The time domain data have an associated sampling frequency defined in (samples/second). The latter relates to the time interval between data points (Dt) as follows:

$$f_s = \frac{1}{Dt} \tag{4-12}$$

For example, if we take 100 regularly spaced measurements, in 0.5 seconds we have a sample of 100 data points (N = 100). The time between samples is found by dividing the total time by the total number of samples. So, if we take a sample every 0.005 seconds (i.e., 0.5 secs / 100 samples) then:

$$Df = \frac{f_s}{N} \tag{4-13}$$

where Df is the frequency resolution that in this example is 2 Hz $\left(= \frac{200\ Hz}{100\ samples}\right)$.

The raw form of the frequency information that FFT delivers is not in a physically meaningful form (e.g., values may be complex numbers). Thus, we need to scale the results to find the amplitude and phase. To identify the amplitude, we need to take the absolute magnitude of the complex number, multiply it by 2 and divide by the total number of points

$$Magnitude = 2\frac{|X_k|}{N} \tag{4-14}$$

where X_k is the complex number at frequency k.

The phase angle information is then determined from taking the tangent of the real and complex parts of the FFT output. This process of FFT is mathematically complex but MATLAB can be used to obtain X_k. Following this, scaling of the magnitude and phase should be done manually. The fastest oscillation (i.e., the highest frequency) depends on how rapidly the data is sampled. Therefore, it can be measured as

$$f_{max} = \frac{N}{2} Df \tag{4-15}$$

which is known as the **Nyquist frequency**.

3.The wave energy spectrum

In a random sea the total energy is a sum of the energies of each of the regular waves comprising the irregular sea and is defined as

$$E_{TOT} = \frac{1}{2} \rho g (\zeta_{a1}^2 + \zeta_{a2}^2 + \cdots + \zeta_{an}^2) \tag{4-16}$$

The importance of wave components (each sinusoidal wave) making up the time history of an irregular wave pattern may be quantified in terms of a wave amplitude energy spectrum known as the wave energy spectrum (see Figure 4-7). The time history of the water elevation is

$$\zeta_t = \bar{\zeta} + \sum_{n=1}^{\infty} \zeta_{i0} \sin(\omega_i t + \varepsilon) \tag{4-17}$$

The area under the curve depicted in Figure 4-7 equals the energy contained in that frequency component.

The spectral ordinate $S_\zeta(\omega_n)$ is the value on the vertical axis of Figure 4-7. Thus, the spectral ordinate for each frequency is

$$S_\zeta(\omega_i) = \frac{\zeta_{i0}^2}{2\delta\omega} \tag{4-18}$$

If the energy spectrum is known, it is possible to reverse the spectral analysis process and generate a corresponding time history by adding a large number of component sine waves as per Eq.(4-17). The measured time history of the water surface elevation and the wave energy spectrum are both representations of the same information – the seaway. The variance is a measure of the degree of spread in the wave surface. Wave amplitudes are a measure of the wave energy. For water elevation the larger the waves the larger the variance and the higher the energy in the seaway. Equation (4-4) depicts the variance m_0. An alternative definition is the variance of the irregular time history is equal to the first statistical moment area under the energy wave spectrum. Thus,

$$m_0 = \int_0^{\infty} S_\zeta(\omega) d\omega \tag{4-19}$$

The wave energy spectrum $S_\zeta(\omega)$ is determined from the wave amplitudes and frequencies from a DFT of the wave time history. Therefore, the wave velocity spectrum $S_{\dot{\zeta}}(\omega)$ and the wave acceleration spectrum $S_{\ddot{\zeta}}(\omega)$ are defined by using the velocity and acceleration amplitudes and frequencies as

$$S_{\dot{\zeta}}(\omega_i) = \omega_i^2 S_\zeta(\omega_i) \quad (4\text{-}20)$$

$$S_{\ddot{\zeta}}(\omega_i) = \omega_i^4 S_\zeta(\omega_i) \quad (4\text{-}21)$$

In general the relationship between a spectral moment and the wave energy curve is

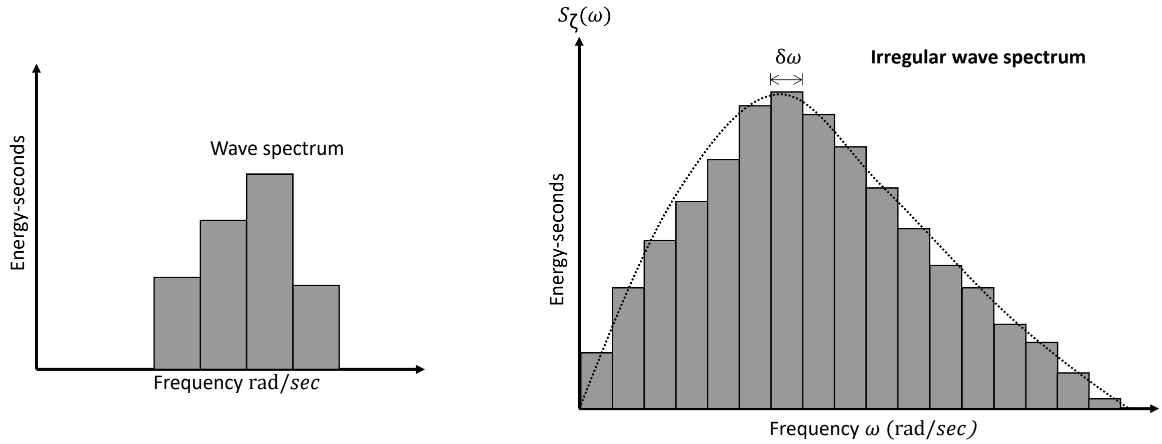
$$m_n = \int_0^\infty \omega^n S_\zeta(\omega) d\omega \quad (4\text{-}22)$$

Thus, the second and fourth-order spectral moments are defined as

$$m_2 = \int_0^\infty \omega^2 S_\zeta(\omega) \quad (4\text{-}23)$$

$$m_4 = \int_0^\infty \omega^4 S_\zeta(\omega) \quad (4\text{-}24)$$

These spectral moments can be used to link the spectra to the statistical characteristics of Table 4-1.



(a)Example of four wave spectrum

(b) Example of irregular wave spectrum

(C) Wave spectrum

Figure 4-7 Wave energy spectrum

Table 4-1 Wave characteristics associated with spectral moments

Mean Frequency	$\bar{\omega} = \frac{m_1}{m_0}$
Mean period	$\bar{T} = \frac{2\pi}{\bar{\omega}} = 2\pi \frac{m_1}{m_0}$
Peak period	$T_P = 2\pi \sqrt{\frac{m_2}{m_4}}$
Zero-crossing period	$T_Z = 2\pi \sqrt{\frac{m_0}{m_2}}$

The **spectrum bandwidth** describes the relative width of the wave energy spectrum compared to the height. **A narrow band spectrum** is concentrated in a narrow range of frequencies and has little or no energy in other frequencies. In a **wide band spectrum,** the energy is distributed among wide range of frequencies. A parameter that may be used to measure the band narrowness of a wave spectrum using the spectrum moments is the **bandwidth parameter** (ε) defined as the ratio between average period of the peaks and the average zero-crossing period according to the equation

$$\varepsilon = \sqrt{1 - \frac{{T_P}^2}{T_z^2}} = \sqrt{1 - \frac{m_2^2}{m_0 m_4}} \tag{4-25}$$

As the bandwidth parameter approaches zero, the spectrum becomes extremely narrow. On the other hand, as the bandwidth parameter approaches one it becomes extremely broad.

4. Ocean wave spectra

When observing the ocean's surface, we see various random waves. The variation in surface elevation over time makes up what in mathematical terms is referred to as *"time series"*. In practice, we convert these signals to the frequency domain or to a spectral representation of the same data. The wave spectrum is much more useful for assessing the vessel's performance than time series data. Naval architects developed mathematical expressions known as idealized wave spectra. These describe the distribution of wave energy with frequency for a specified wave height and period.

Sea waves are primarily the result of wind transferring energy to the sea surface (see Lecture 3). The kinetic energy of the wind (wind speed) creates potential energy of the water (waves). The height and length of the generated waves depend on the wind velocity, the length of time wind blows over the water surface and the fetch; i.e. the distance of water over which the wind blows before reaching the land. Open oceans have infinite fetch. Bays, lakes and coastal areas are limited in fetch and encompass higher frequency waves that are shorter and steeper. If there are no fetch limitations waves reach an equilibrium where the amount of energy transferred from the wind maintains the wave heights. However, the dissipation of the water (viscous and wave breaking) prevents additional amplitude growth. A sea at this condition is known as fully developed.

For ship design purposes we use different formulae to represent open ocean and coastal (limited fetch) wave conditions. The first ocean spectra model considered is the ISSC (International Ship as and Offshore Structures Congress) one parameter spectrum originally introduced by Pierson and Moskowitz (1964). The only input of this wave spectrum is wind speed. The spectrum assumes there is plenty of area for the wind/water interface and the wind has been blowing for long enough so that the wave field has reached a state of equilibrium. The derivation of this spectrum has been based on extensive measurements in the North Atlantic Ocean and it is intended to represent the spectrum of fully developed seas as follows:

$$S_\zeta(\omega) = \frac{0.0081 g^2}{\omega^5} e^{-0.74\left(\frac{g}{W_{19.5}\,\omega}\right)^4} \tag{4-26}$$

where $W_{19.5}$ is the average wind speed in m/s at 19.5 m above the sea surface. The units for the spectrum are m^2/s. Since 19.5 m is not a typical height for wind measurements this can be related to the more standard 10 m height by the empirical adjustment $W_{19.5} = 1.026\, W_{10}$ (see Figure 4-8).

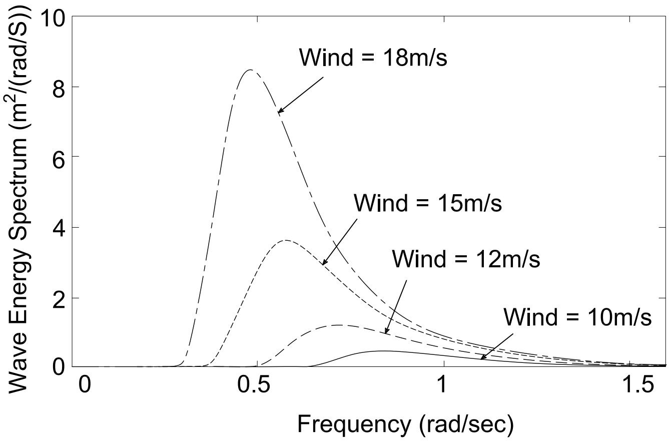


Figure 4-8 Example Pierson Moskowitz spectra for different wind speeds

Another open ocean wave spectrum is the **ITTC** (International Towing Tank Conference) two-paramter spectrum introduced by Bretschneider (1973)**.** This spectrum depends on the given significant wave height and the modal period (i.e. the period that coincides with the peak of the wave energy spectrum) as follows:

$$S_\zeta(\omega) = \frac{1.25}{4}\left(\frac{\omega_0}{\omega}\right)^4 \frac{H_{1/3}^2}{\omega} e^{-1.25\left(\frac{\omega_0}{\omega}\right)^4} \tag{4-27}$$

where $H_{1/3}$ is the significant wave height, ω_0 is the modal wave frequency $\omega_0 = (\frac{2\pi}{T_0})$, where T_0 is the modal wave period). Figure 4-9 shows various ITTC wave spectra for different wind speeds.

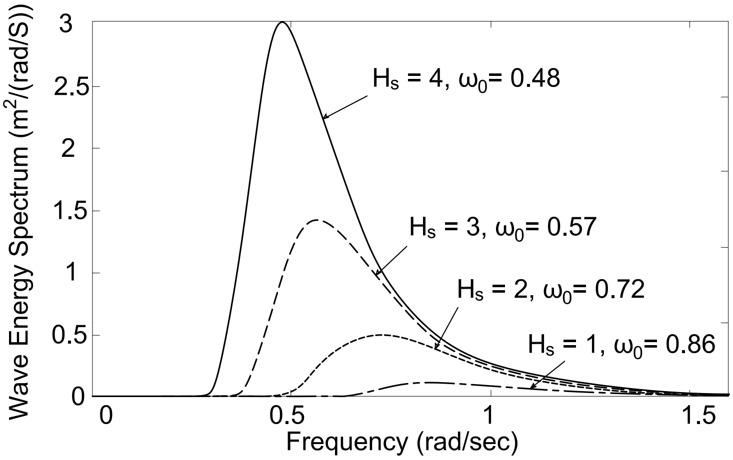


Figure 4-9 Example of ITTC (Bretschneider) spectra for different significant wave height / modal period combinations.

For limited fetch conditions, we typically use the Jonswap (**J**oint **N**orth **S**ea **W**ave **O**bservation **P**roject) spectrum. This is intended to represent open wave conditions with limited fetch (e.g., North Atlantic). The spectrum is narrower, i.e. has higher peaks than the pure open ocean spectra. It is a **three-parameter spectrum** and accordingly considers wave speed, fetch and steepness (γ) as follows:

$$S_\zeta(\omega) = \frac{\alpha g^2}{\omega^5} e^{-\left(\frac{5\omega_0^4}{4\omega^4}\right)} \gamma^r \quad (4\text{-}28)$$

In the above equation :

- Factor α depends on the wind speed W_{10} and fetch F and is defined as

$$\alpha = 0.076 \left(\frac{W_{10}^2}{Fg}\right)^{0.22} \quad (4\text{-}29)$$

- The modal wave frequency is defined as

$$\omega_0 = 22 \left(\frac{g^2}{W_{10} F}\right)^{1/3} \quad (4\text{-}30)$$

- The factor r depends on wavefrequencies and the spectral ordinate parameter σ defined as

$$r = e^{-\left[\frac{(\omega-\omega_0)^2}{2\sigma^2\omega_0^2}\right]} \quad (4\text{-}31)$$

$$\text{for } \sigma = \begin{cases} 0.07 \; if & \omega \le \omega_0 \\ 0.09 \; if & \omega > \omega_0 \end{cases} \quad (4\text{-}32)$$

It is noted that the *"steepness factor - γ "* has the typical value of 3.3 but may range from 1 to 7 depending on the sea conditions. The Jonswap spectrum can also be expressed as a function of significant wave height, modal frequency and γ as

$$S_\zeta(\omega) = B_j \overline{H}_{1/3}^2 \frac{2\pi}{\omega}\left(\frac{\omega_0}{\omega}\right)^4 e^{-\left[\frac{5}{4}\left(\frac{\omega_0}{\omega}\right)^4\right]}\gamma^r \tag{4-33}$$

$$\text{where } B_J = \frac{0.06238}{0.230+0.0336\gamma-\frac{0.0185}{1.9+\gamma}}\left[1.094-0.01915 ln\gamma\right] \tag{4-34}$$

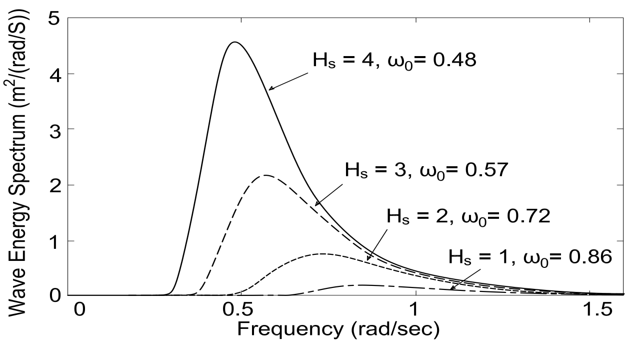


Figure 4-10 Example of Jonswap spectra for different wind speed and fetch combinations.

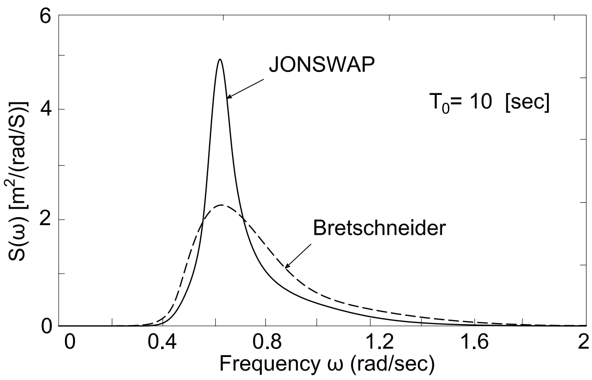


Figure 4-11 *Comparison of steepness difference between Jonswap* & Bretschneider spectra for $H_{1/3} = 4m$

5. Directional Spreading

The typical wave spectra described in section 4 assume a single wave heading. However, typical wind wave spectra have a directional spread related with the direction of the wind. Large spreading implies that the waves are confused and short crested. Narrow spreading implies that the waves are long crested or unidirectional. Waves produced by swell are almost long crested. This is because since the crests of the wave system observed outside the storm/fetch area become nearly parallel. Therefore, in confused seas we can say that wave energy at a given location has an angular distribution as well as a distribution over a range of frequencies.

Spectral models without spreading can over-predict the significant wave height. Therefore, in some applications, especially in fatigue analysis, we use directional spreading spectrum represented in terms of unidirectional wave spectra as follows

$$S(\omega,\theta) = S(\omega)D(\theta,\omega) = S(\omega)D(\theta) \tag{4-35}$$

In the above equation, the directional spreading function $D(\theta,\omega)$ is usually simplified to $D(\theta)$ and θ is the angle between the direction of elementary wave trains and the main wave direction of the short-crested wave system. The spreading function $D(\theta,\omega)$ is a non-negative and dimensionless function integrated in the interval $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$:

$$\int_{-\pi/2}^{\pi/2} D(\theta,\omega)\mathrm{d}\theta = 1 \tag{4-36}$$

Various different parametric models can be used to represent the directional spreading (e.g. Wrapped normal, sech-2, Poisson, etc.). The cos^{2s} function, equation (4-37), is usually used for practical reasons. This model assumes that the directional spectrum is a maximum in the wave dominant direction θ_0 and gradually decreases as the angle to the wind direction increases. It can be noticed that the function is symmetrical with respect to the mean propagation direction θ_0 and independent of the wave frequency ω.

$$D(\theta) = G(s)\cos^{2s}(\theta-\theta_0) \tag{4-37}$$

In the above equation, $G(s)$ is a normalization factor given by:

$$G(s) = \frac{2^{2s-1}}{\pi}\frac{\{\Gamma(s+1)\}^2}{\Gamma(2s+1)} \tag{4-38}$$

where Γ is the Gamma function and s is the spreading factor which is a positive real number that controls the degree of concentration of the spreading energy around θ_0. Figure 4-12(b) presents an illustration of the wave directional spectrum, using cosine squared spreading over $\pm$90°, at discrete heading intervals of 15°. Figure 4-12(b) depicts the spreading function $D(\theta)$ for various values of the spreading parameter s. As the spreading parameter value increases, more energy is concentrated around dominant wave direction θ_0. In the limit case when s asymptotically tends to infinity, the spreading function eventually approaches the unidirectional case.

(a) (b)

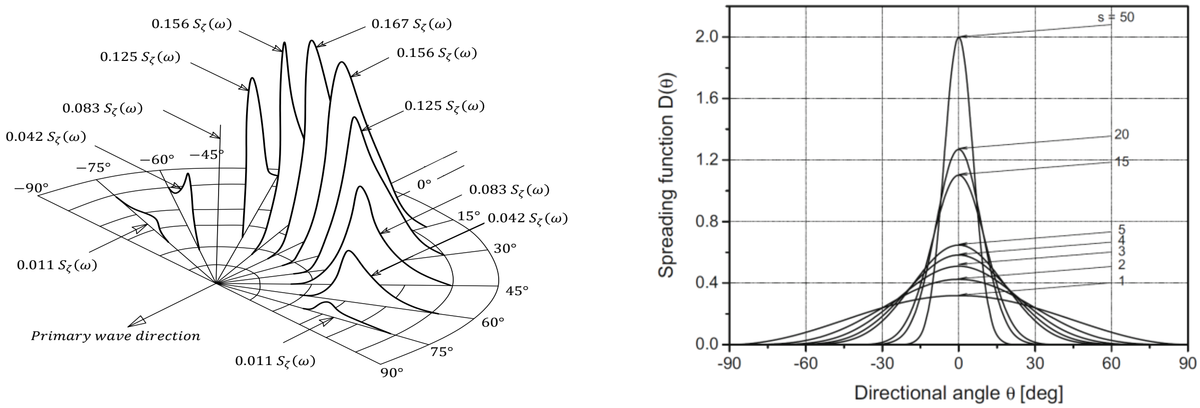


Figure 4-12 Representation of the directional spreading function ©Marcelo Caire (a) Representation of the directional spectrum at discrete heading intervals of 15°; (b) cosine squared spreading over $\pm 90°$.

6. The statistics of sea states

We have discussed in the first sections of this lecture how the spectrum can be idealized using some wave characteristics corresponding to wave records or measurements such as the significant wave height and period. In open seas, these characteristics vary depending on the area of operation. Naval architects need to choose specific values of the wave characteristics in different locations and seasons. A summary of these data is included in the **BMT Atlas of Global Wave Statistics** (Hogben et al., 1986).

The BMT wave data statistics are divided into different numbered regions as shown in Figure 4-13. They are subdivided into different wave directions and presented in the form of scatter diagrams, giving the joint probability of occurrence for combinations of significant wave height and zero-crossing periods (see Table 4-2). For instance, the probability of occurrence of wave heights from all directions in the range of 6 – 7 meters for periods that range between 9 and 10 seconds is 32/1,000 = 0,032. The right hand of the wave diagram is the probability of occurrence of each significant wave height range for all wave periods (see Table 4-2), while the probability of occurrence of each significant wave period range for all wave periods is shown in the bottom of each diagram. North Atlantic especially in the winter season and the North Sea are the most critical areas. Global wave statistics may deviate from modern satellite measurements because (1) they were formed in the 1960s and since then due to global warming the environmental conditions changed; (2) they are based on visual observations. Table 4-3 illustrates the range of wind speed, significant wave height, modal wave period and the probability of each sea state number in the North Atlantic. Sea state number is a description of the sea state proposed by the World Meteorological Organization (WMO), usually used in oceanography. Some of the basics on the importance of wave distributions on the design of ships and offshore structures are given in Ochi (1978) and will be revisited at a later stage in this course.

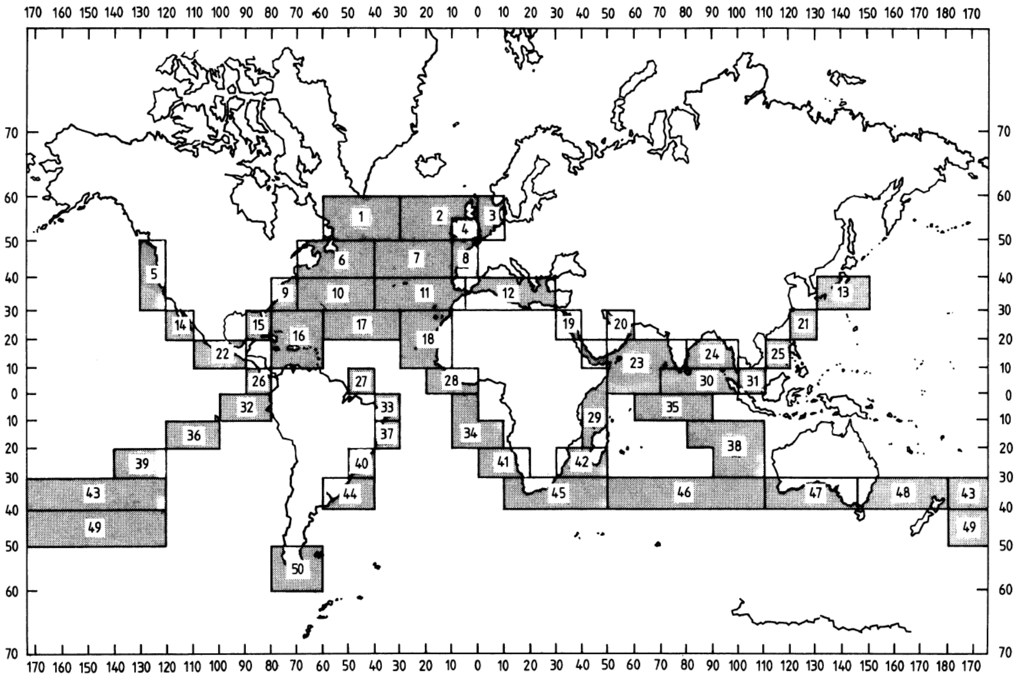


Figure 4-13 Areas for which homogenous waves statistics are available (Rawson and Tupper, 2001)

Table 4-2 10^4 probability of combinations of H_s and T_m for all wave directions and months at a position in the North Atlantic (Hogben, Dacunha and Olliver 1986)

$H_s[m]$ $T_m[s]$	0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9	9-10	10-11	11-12	12-13	13-14	14-33	Sum
$0-3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$3-4$	4	22	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$4-5$	24	501	123	1	0	0	0	0	0	0	0	0	0	0	0	0
$5-6$	65	725	902	224	7	0	0	0	0	0	0	0	0	0	0	0
$6-7$	24	702	697	815	384	31	1	0	0	0	0	0	0	0	0	2655
$7-8$	1	192	579	484	513	428	113	8	0	0	0	0	0	0	0	2318
$8-9$	0	14	195	365	265	263	266	145	28	2	0	0	0	0	0	1543
$9-10$	0	0	11	90	151	115	76	95	90	35	5	0	0	0	0	669
$10-11$	0	0	0	3	22	42	30	18	18	25	19	7	1	0	0	186
$11-12$	0	0	0	0	0	3	7	5	2	2	3	5	4	1	0	31
$12-13$	0	0	0	0	0	0	0	1	1	0	0	0	0	0	1	3
Sum	117	2156	2507	1982	1342	881	493	273	138	63	27	12	5	2	1	10000

Table 4-3 Sea State Numeral Table for the Open Ocean North Atlantic

Sea State Number	Significant Wave Height (ft)		Sustained Wind Speed (Kts)		Percentage Probability of Sea State	Modal Wave Period (s)	
	Range	Mean	Range	Mean		Range	Most Probable
0-1	0-0.3	0.2	0-6	3	0	-	-
2	0.3-1.5	1.0	7-10	8.5	7.2	3.3-12.8	7.5
3	1.5-4	2.9	11-16	13.5	22.4	5.0-14.8	7.5
4	4-8	6.2	17-21	19	28.7	6.1-15.2	8.8
5	8-13	10.7	22-27	24.5	15.5	8.3-15.5	9.7
6	13-20	16.4	28-47	37.5	18.7	9.8-16.2	12.4
7	20-30	24.6	48-55	51.5	6.1	11.8-18.5	15.0
8	30-45	37.7	56-63	59.5	1.2	14.2-18.6	16.4
>8	>45	>45	>63	>63	<0.05	15.7-23.7	20.0

7. Extreme waves

Freak waves (or rogue or abnormal waves) are an open water phenomenon, in which winds, currents, non-linear phenomena such as solitons, and other undefined circumstances cause a wave to briefly form a far larger wave. They are considered rare but potentially very dangerous, since they can involve the spontaneous formation of massive waves far beyond the usual expectations of ship designers, and therefore can overwhelm the usual capabilities of oceangoing vessels which are not designed for such encounters. The underlying physics that makes phenomena such as rogue waves possible is that different waves can travel at different speeds, and so they can *"pile-up"* in certain circumstances.

In deep ocean conditions the speed of a gravity wave is proportional to the square root of its wavelength, i.e., the distance of peak-to-peak between adjacent waves. Other situations can also give rise to rogue waves, particularly situations where non-linear effects or instability effects can cause energy to move between waves and be concentrated in one or very few extremely large waves before returning to ''*normal*'' conditions. Eyewitness accounts from mariners and damage inflicted on ships have long suggested that they occur more frequently than originally thought. The first scientific evidence of their existence came with the recording of a rogue wave by the "Gorm platform" in the central North Sea in 1984. A stand-out wave with height of 11 metres emerged in a relatively low sea state conditions. However, what caught the attention of the scientific community was the digital measurement of a rogue wave at the Draupner platform in the North Sea on January 1, 1995. This is known as the **Draupner wave** corresponding to the maximum wave height of 25.6 metres (McAllister, 2019). During that event, a minor damage was inflicted on the platform far above sea level. Since then, the existence of freak waves has been confirmed by video and photographs, satellite imagery, radar of the ocean surface, stereo wave imaging systems, pressure transducers on the seafloor, and oceanographic research vessels. In February 2000, the British oceanographic research vessel, RRS Discovery, was sailing in the Rockall Trough west of Scotland. She encountered the largest waves ever recorded by any scientific instruments in the open ocean, with a significant wave height of 18.5 meters and individual waves up to 29.1 metres. In 2004 scientists using three weeks of radar images from European Space Agency satellites found ten rogue waves, each 25 meters or higher.

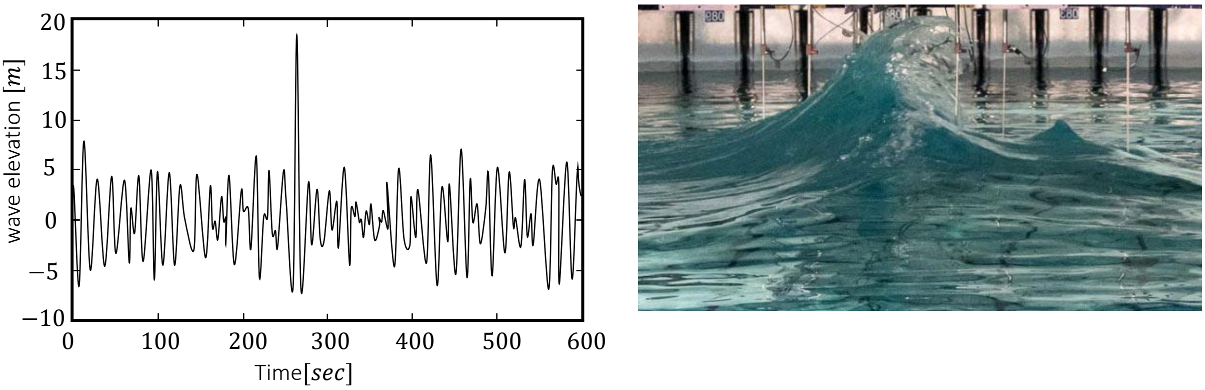


Figure 4-14 Wave height as a function of time recorded on New year's Day, 1995, using a radar pulse-echo system setup on the Draupner oil rig in the North Sea off Norway (left). reconstruction of the Draupner wave (right), (McAllister et al. 2019).

A rogue wave is a natural ocean phenomenon that is not caused by land movement, only lasts briefly, occurs in a limited location, and most often happens far out at sea. Rogue waves are, therefore distinct from **tsunamis** that are caused by a massive displacement of water, often resulting from sudden movements of the ocean floor, after which they propagate at high speed over a wide area. They are nearly unnoticeable in deep water and only become dangerous as they approach the shoreline and the ocean floor becomes shallower. Tsunamis do not present a threat to shipping at sea. They are also distinct from (a) mega-tsunamis, which are single massive waves caused by sudden impact, such as meteor impact or landslides within enclosed or limited bodies of water and (b) the so called **hundred-year wave** which is a purely statistical prediction of the highest wave likely to occur in a hundred-year period in a particular body of water.

Figure 4-15 Waves approach Miyako City after a 9.0 magnitude earthquake hit Japan. This tsunami led to more than 15,000 deaths (Image credit: NBC News 4.12.2018)

The linear sea wave models introduced in the previous sections rely on that, the wave state is a stationary process, the wind velocity is constant without statistical variability and the distance over which the waves develop and the duration for which the wind blows are sufficient for the waves to achieve their maximum energy. In these models, the sea waves are represented by sinusoidal forms that vary in a regular way around an average wave height. Therefore, these models can be used to predict the wave height only on a statistical basis, i.e. wave height conforms to a Rayleigh distribution, see equation (4-8) and Figure 4-4. However, freak waves of up to 35m in height are much more common than this probability theory could predict. Therefore, these models cannot account for the existence of freak waves that have been observed and measured with increasing regularity throughout the oceans.

There are three principal categories of a range of freak wave phenomena that have yet to be fully classified namely

- **Walls of water** travelling up to 10 km over the ocean surface before become extinct
- **Three sisters** are groups of three waves
- **Single, giant storm waves** that build up to more than four times the average height of storm waves, but they collapse in a relatively short time

One of the most common models used to explain the occurrence of the freak wave in deep water is the Nonlinear Schrödinger equation **NLS**. The equation describes freak waves that are caused by the nonlinear effect in which the energy of many randomly generated waves is combined into a single wave front. This wave continues to grow until collapsing under its own weight. The freak waves that are caused by the **diffraction effect** and **current focusing** cannot be described by the NLS. The **diffraction** effect occurs when a collection of small diffracting waves (its pattern is affected by the angle of incidence and/or seabed) coherently combine in phase to produce a freak wave. While the **current focusing** occurs due to the interference between the storm force waves that are driven from opposite directions. The most commonly quoted nonlinear form of Schrödinger equation is the cubic NLS (Chiron, 2012)

$$(\nabla^2 + i\,\partial_t)\Psi = \kappa|\Psi(\mathbf{r},t)|^2\Psi(\mathbf{r},t) + s(\mathbf{r},t), \qquad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad (4\text{-}39)$$

where Ψ is the wave function, which is expressed by wave period t and space $\mathbf{r} \equiv (x, y, z)$, and $s(\mathbf{r}, t)$ is the source function. Hence the equation represents a model in which a wave interacts with its own energy $|\Psi|^2$. The coefficient κ determines the strength of this *"self-interaction"* which may be generalized to be $\pm\kappa$. It is noteworthy that all classes of NLS are phenomenological in origin, and their results can be confirmed experimentally, which is the only justification of the validity of these equations.

8. Questions

1. The wave height characteristics from wave records are shown in the table below. Find the values for (a) the total number of waves measured; (b) the average wave height; (c) the significant wave height; (d) the average one-tenth and one-hundredth highest waves.

Wave height	Number of waves
1 m	4
2 m	40
3 m	31
4 m	25
5 m	2

2. Briefly explain what a Wave Energy Spectrum is, where it comes from, and how it can be related to a Rayleigh Probability Distribution.
3. List the three models for ocean wave spectra and identify what parameters they use and what ocean situations they are the most suited to model.
4. What is the relationship between the variance of the irregular wave signal and the wave energy spectrum curve?
5. Which idealized wave energy spectrum would be best to use to describe an open ocean sea state with a given significant wave height and modal period?
6. Plot Bretschneider's wave spectrum of a sea state characterized by significant wave height H_S= 5m and modal period T_0 = 13 seconds and then calculate:
   a. 0^{th},1^{st}, 2^{nd},3^{rd} and 4^{th} spectral moments and wave variance, what is the relation between them? Use Simpson's rule
   b. Spectrum bandwidth
   c. Mean zero up-crossing, mean wave frequency, mean peak period and Standard deviation.
7. Draw an irregular wave in time domain that comprises of the following regular waves:

Wave index	Period T	Amplitude a
1	2.5 sec	1 m
2	6 sec	7.5 m
3	15.5 sec	13 m

8. Use Fast Fourier Transformation function in MATLAB to obtain the regular wave components of the irregular wave in question 7, see the MATLAB script in the tutorial.
9. What are the freak waves? How they are therefore distinct from ''tsunamis''?
10. Why tsunamis waves do not affect ships in deep seas and they present a threat to shipping at shallow water?

9. References

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