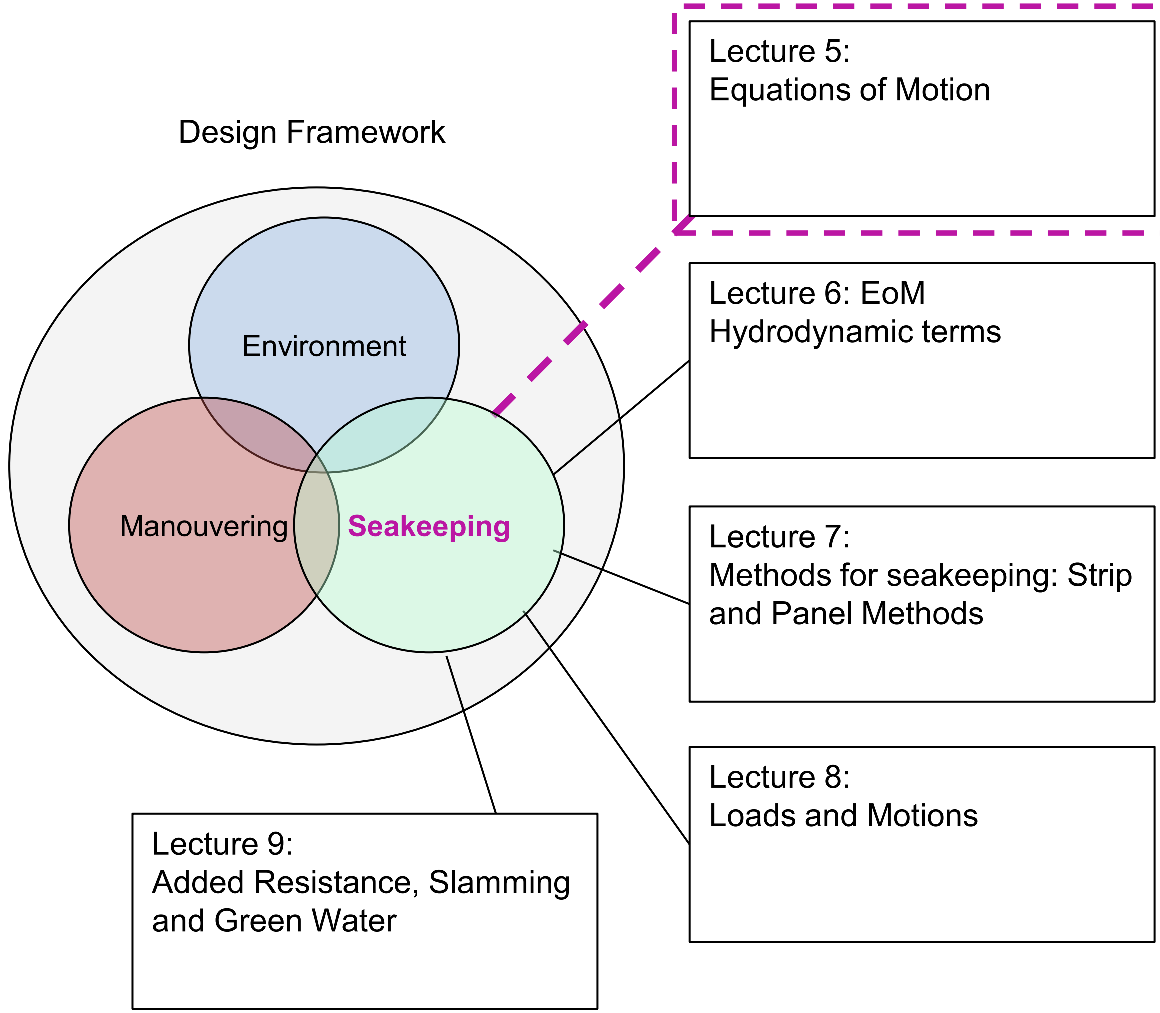
Aalto University
School of Engineering

MEC-E2004 Ship Dynamics (L)

Lecture 5 – Equations of Motion (Part I)

Aalto University
School of Engineering

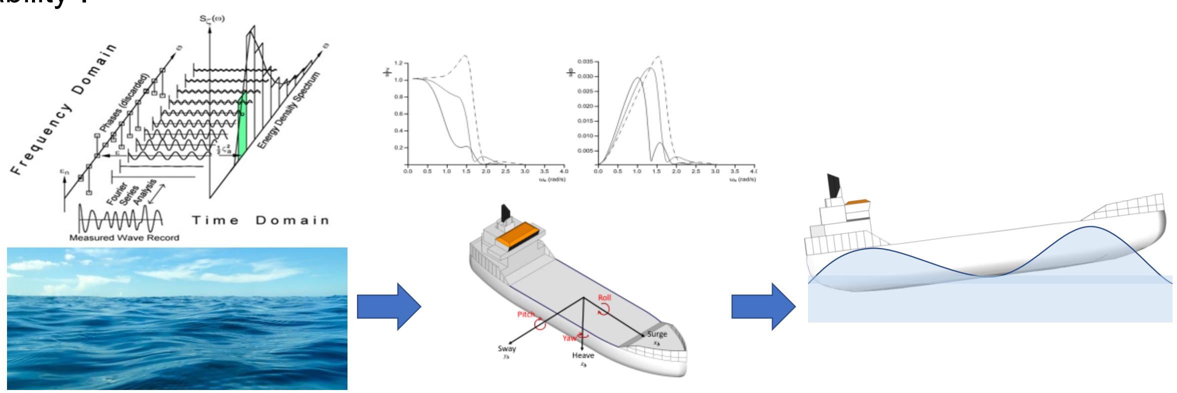
Where is this lecture on the course?



Contents

- **Aim :** To introduce the equations of motion and how these are formed using basic rigid body dynamics.

- <u>Literature</u>
  - ➢ Journee, J.M.J., ”Introduction to Ship Hydromechanics”
  - ➢ Lloyd, A.R.J.M, ”Seakeeping – Ship Behavior in Rough Weather”, John Wiley & Sons
  - ➢ Bertram, V., ”Practical Ship Hydrodynamics”, Butterworth-Heinemann, Ch. 4.
  - ➢ Matusiak, J., ”Ship Dynamics”, Aalto University
  - ➢ Lewis, E. V. Principles of Naval Architecture. Vol. 3, ”Motions in waves and controllability”
  - ➢ Rawson, K. J., ”Basic Ship Theory. Volume 2, Ship dynamics and design - ch.12 Seakeeping & ch.13 Manoeuvrability”.



(a) Input
Waves and Winds

(b) System
RAOs

(c) Output
Motions and structural loads

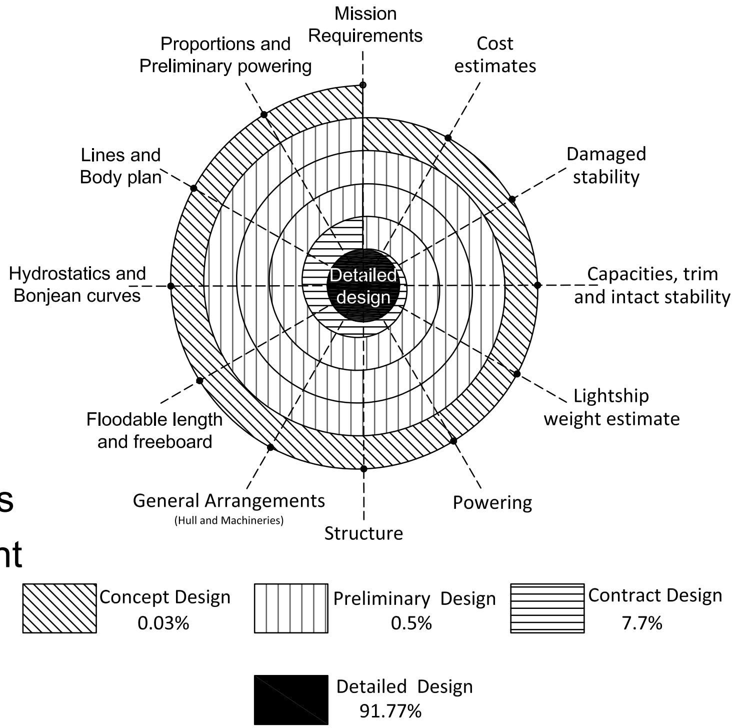
Motivation

- Ship motions are affected by numerous factors such as :
  - Sea state
  - Propulsive equipment (rudders, propulsors etc.)
  - Cargo movement
  - Special general arrangement features

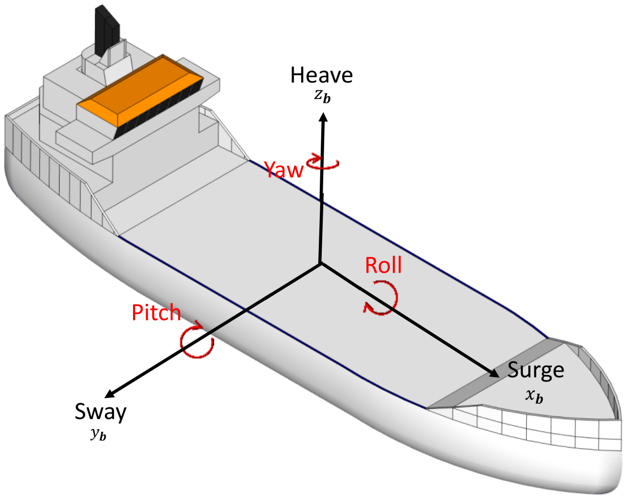
- Practical and well validated methods and procedures that are suitable for ship design are essential.

- Classic methods are based on linear ship dynamics (potential flow analysis methods).
  - They allow us to use spectral techniques and statistics
  - They can be updated with correction factors to account non-linearities

- Non linear methods become useful when ship motions are excessive or we model extreme events. Approaches exist in time-domain for specific sea states, time-frames and using different time histories. CFD approaches also emerge.



Assignment 3



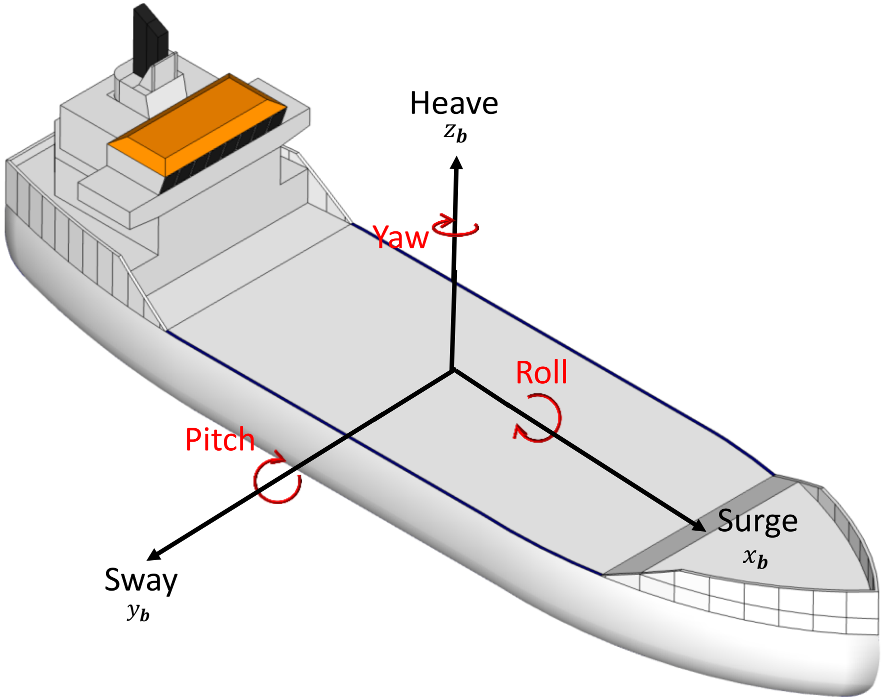
- **Grades 1-3:**
  - ✓ Select a book-chapter related to the ship equations of motion and read it
  - ✓ Identify the main components associated to equations of motion of your ship. How and why they relate with the ship's mission (think in operational safety terms) ?
  - ✓ Discuss how the general arrangement, hull form and operational profile of your ship affect the equations of motion (think in design for safety terms).
  - ✓ Start getting familiar with motions and loads design software (e.g. Maxsurf, Napa, etc.) and reflect the software use to the theory learned

- **Grades 4-5:**
  - ✓ Read 1-2 scientific journal articles related to Ship Equations of Motion
  - ✓ Reflect these in relation to knowledge from books and lecture slides

- Report and discuss the work.

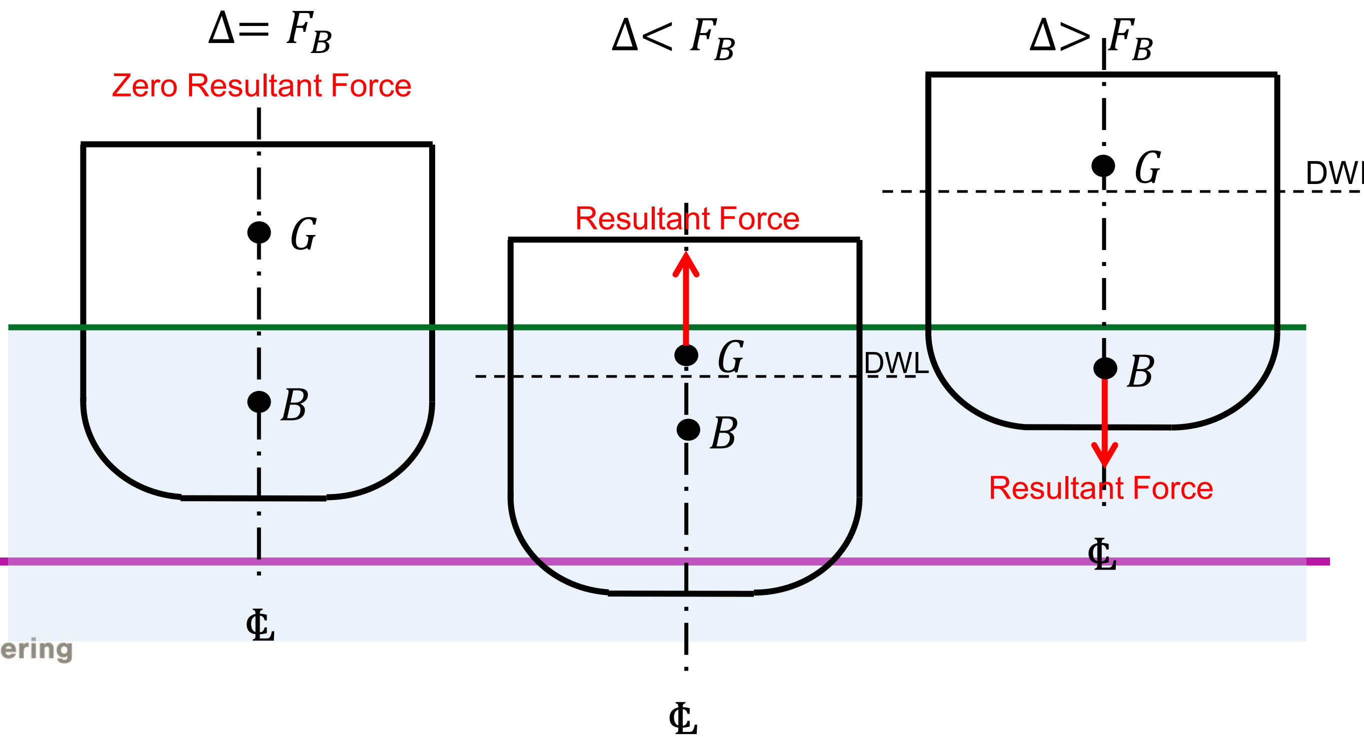
Ship Motions - Introduction

- A rigid ship moves in waves in 6 degrees of freedom (DOF)
- This means that for arbitrarily-shaped ship we will have
  - ✓ 6 equations of motion
  - ✓ 6 unknowns
- These must be solved simultaneously
- For port-starboard-symmetry these equations reduce to two sets of uncoupled EoM containing 3 unknows namely :
  - ✓ surge, heave, pitch
  - ✓ sway, yaw, roll
- We approximate the response by superposition of elementary waves progressing in :
  - ✓ Different lengths
  - ✓ Different directions



Ship Motions - Introduction

- Heave is a rigid body response proportional to the distance displaced (Linear seakeeping).
- Disparity between displacement and buoyancy forces may be considered linear for different waterlines.
- Ships with large water plane area have large heave restoring forces.
  - ✓ *"Beamy" ships (tugs, fishing vessels) suffer short period heave oscillations and high heave accelerations.*
  - ✓ *Ships with small water plane areas will have much longer heave periods and experience lower heave accelerations, so it is more comfortable.*



Encounter frequency

- For ships moving with forward speed we use the encounter frequency (ω_e) instead of the absolute wave frequency (ω)

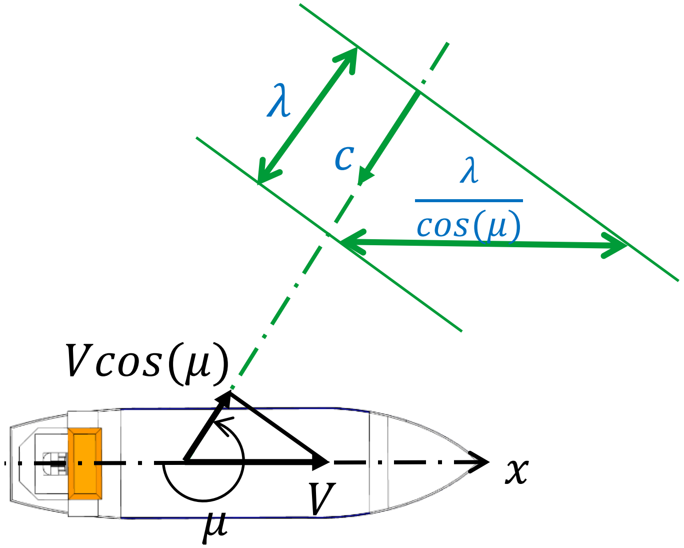
- The encounter frequency and the ship dynamics depend on whether she is advancing into the waves or travelling in their direction.

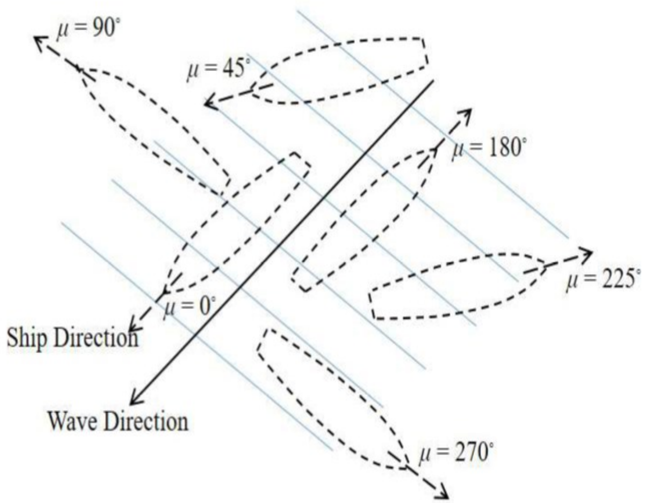
- The encounter period

$$T_E = \frac{\lambda}{(c - U\cos(\mu))}$$

- The encounter frequency

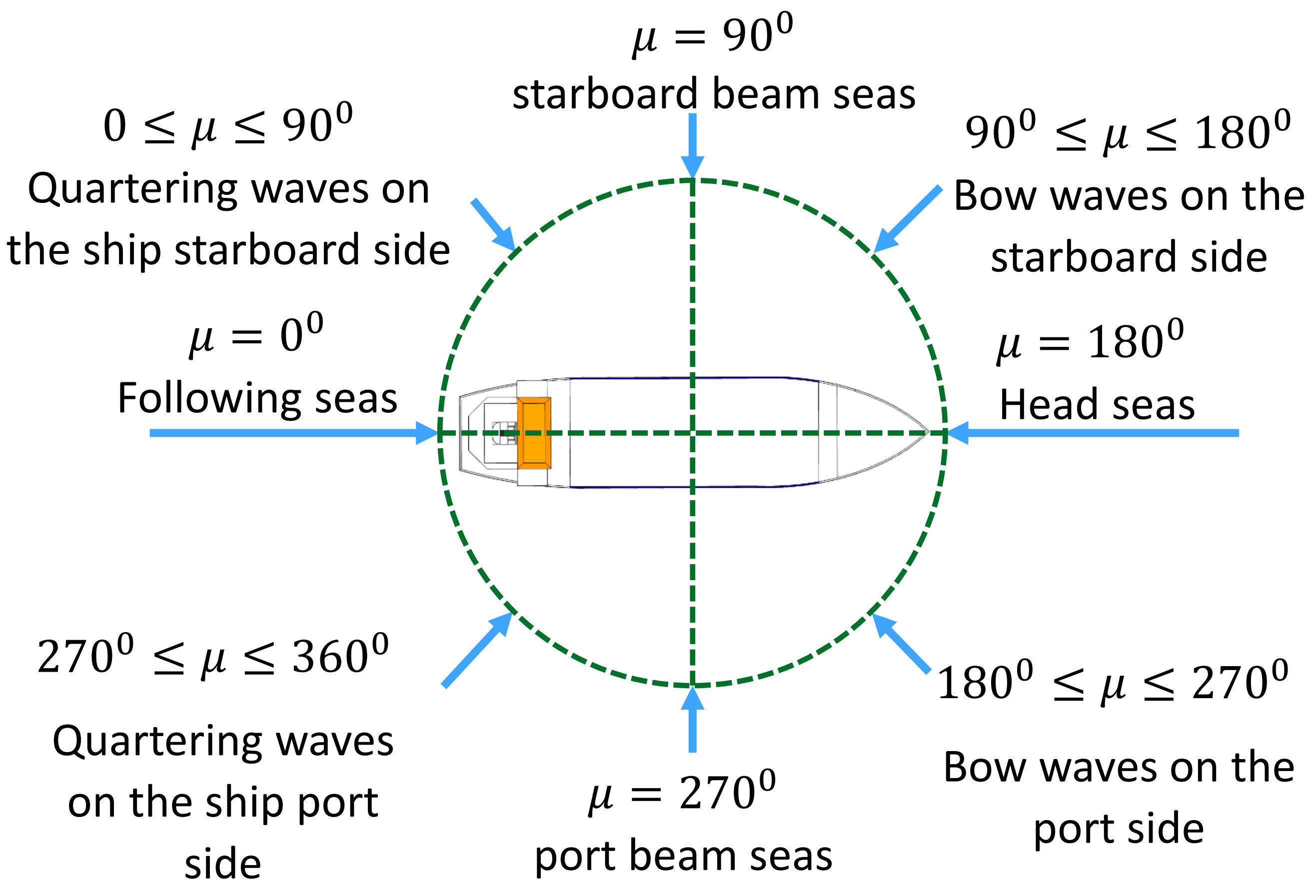
$$\omega_e = \frac{2\pi}{T_E} = \omega - \frac{\omega^2}{g} U\cos(\mu)$$





Definition of headings

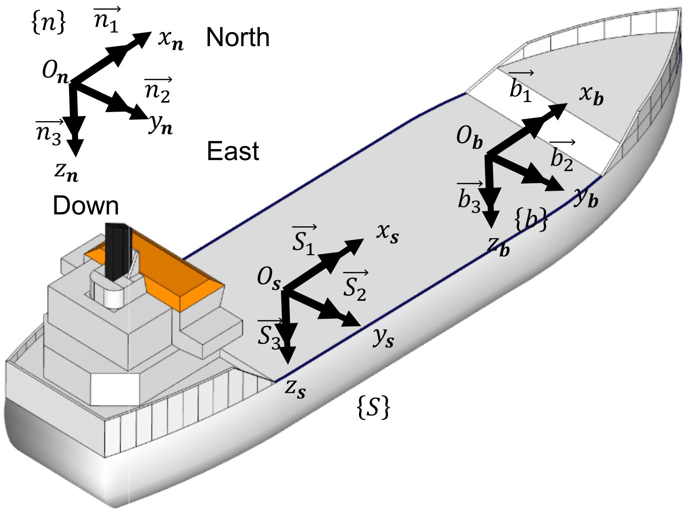
- **The heading angle determines the "type" of seas the ship experiences.**



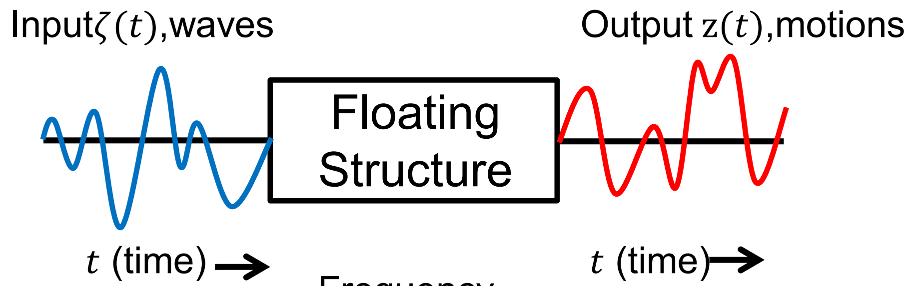
Coordinate system

Describes the position and orientation of a ship

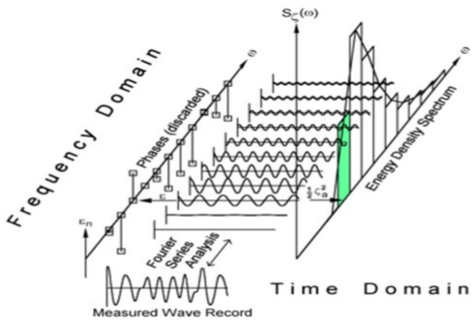
- **Earth fixed inertial coordinate system {n} -** *Defines the position of the vessel on the earth, the direction of wind, waves and current.*
- **Body-fixed coordinate system (O_b) -** Expresses velocity and acceleration measurements taken onboard
- **Seakeeping coordinate system**
  - *Positioned at the center of gravity*
  - *Moves with the ship*
  - *Sensitive to wave elevation and the definition of hydrodynamic forces*

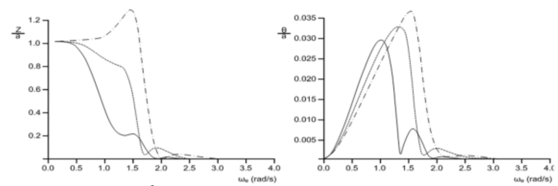


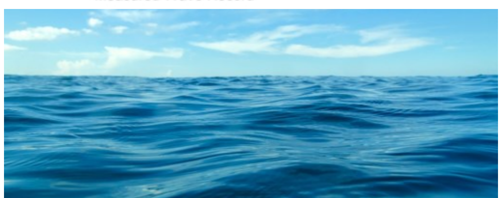
Response Spectrum

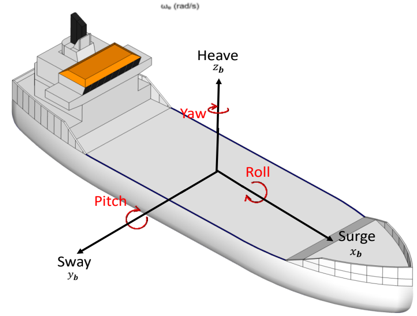


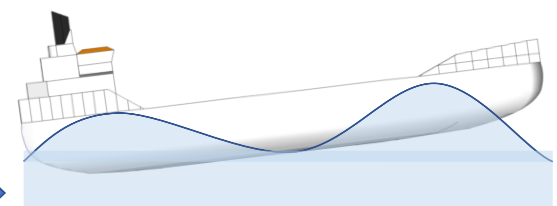
Wave spectrum → Frequency characteristics RAO → Motion spectrum

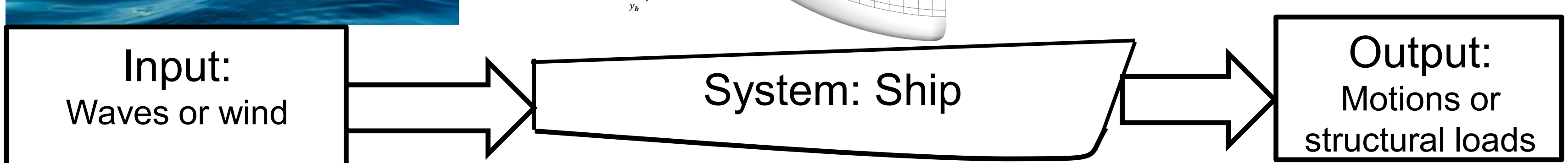




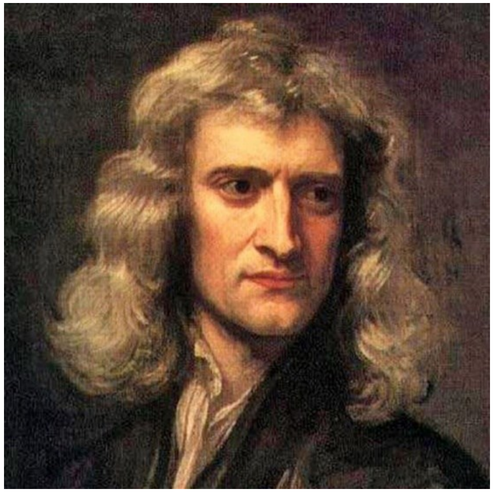




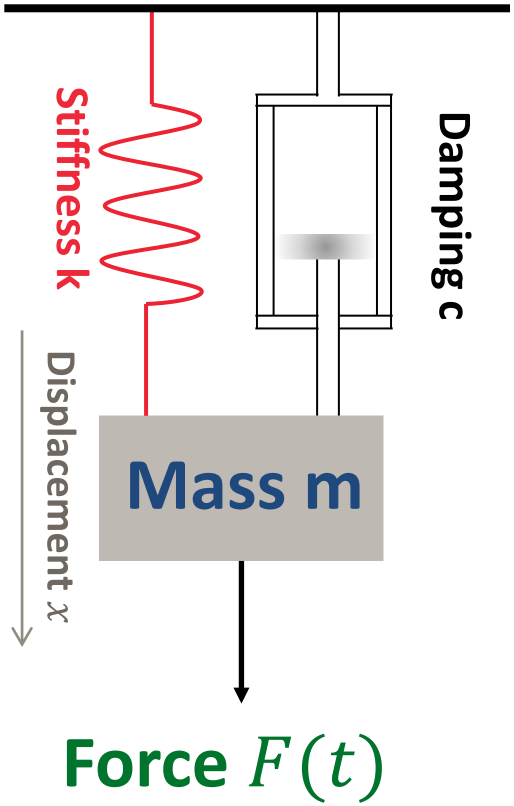




Dynamics of ships as rigid bodies



$$\sum \vec{F} = m\ddot{x}$$



Newton's 2nd Law

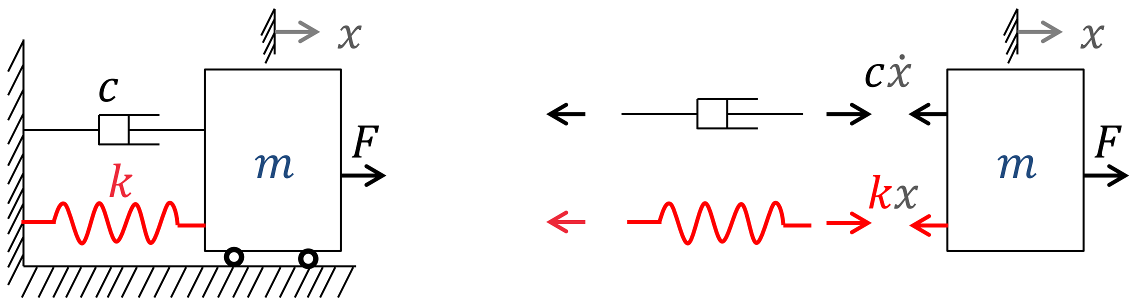
$$m\ddot{x} + c\dot{x} + kx = F(t)$$

- If we set $F(t) = 0$ then we obtain the complementary function; i.e. the function expressing the response of the system when we have free vibration

- F(t) is also known as the particular integral ; i.e. a function expressing the excitation and affecting the frequency response function

- In ship dynamic terms this means that dynamic response may be simply affected by the complementary function or her combination with the particular integral

- For ships the mass (m), stiffness (k) and damping (c) terms should include both wet and dry parts



Case 1 : Undamped free vibration (1 - DOF)

- Assume the system is conservative and the vibration is free. The equation of motion reduces to:

$$m\ddot{x} + kx = 0$$

- Assume sinusoidal solution $x = e^{\lambda t}$

Natural frequency of the system

$$\lambda^2 m + k = 0,\ \lambda = \pm\sqrt{\frac{k}{m}} = \pm j\omega_n$$

- The response is defined as :

$$x = A_1 e^{j\omega_n t} + B_1 e^{-j\omega_n t} = A\, sin(\omega_n t) + B\, cos(\omega_n t) = X\, sin(\omega_n t + \phi)$$

- The amplitude and phase are defined as :

$$X = \sqrt{A^2 + B^2} \qquad \phi = \tan^{-1}(B / A)$$

Case 1 : Undamped free vibration (1 dof)

- To find X and ϕ, we need the following initial conditions:

$$x(t=0) = x_0 = X\sin\phi \text{ and } \dot{x}(t=0) = v_0 = X\,\omega_n\cos\phi$$

- Solving this system of two equations and two unknowns gives us:

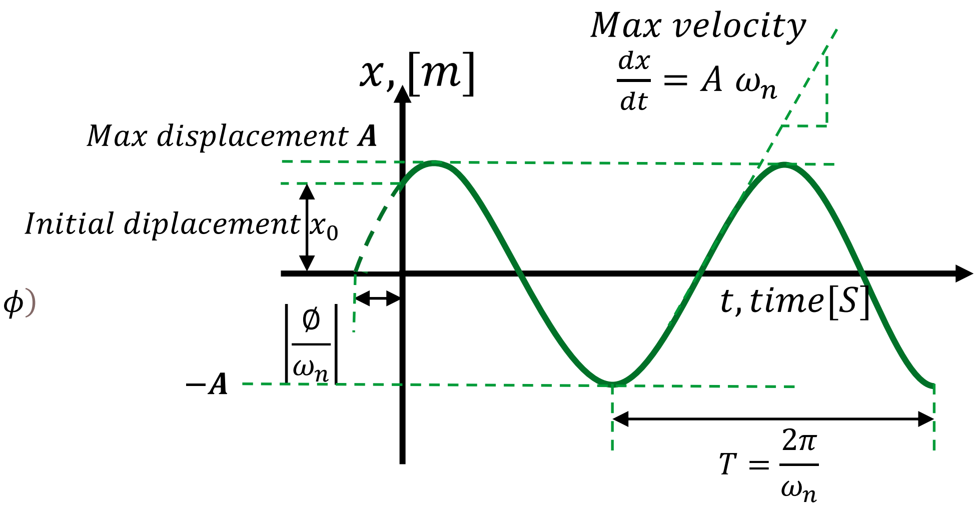
$$X = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \qquad \phi = tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right)$$

- Therefore the final solution of this system displacement, velocity and acceleration are defined as:

$$x(t) = X\sin(\omega_n t + \phi) = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2}\, si\,n(\omega_n t + \phi)$$

$$\dot{x}(t) = X\omega_n\cos(\omega_n t + \phi) = \sqrt{\omega_n^2 x_0^2 + v_0^2}\cos(\omega_n t + \phi)$$

$$\ddot{x}(t) = -X\omega_n^2\sin(\omega_n t + \phi) = -\omega_n\sqrt{\omega_n^2 x_0^2 + v_0^2}\sin(\omega_n t + \phi)$$



Case 2 : Damped free vibration (1- DOF)

- The amplitude of oscillation of the spring, mass, damper system will reduce with time due to damping effects. The damper works by dissipating the energy of the system to zero. For this case Newton's equation becomes :

$$m\ddot{x} + c\dot{x} + kx = 0$$

- Assume sinusoidal solution $x = e^{\lambda t}$

$$m\lambda^2 + c\lambda + k = 0, \qquad \lambda_{1,2} = \frac{-c + \sqrt{c^2 - 4mk}}{2m}$$

- There are three solutions to the above differential equation that link to three different types of motions:

1. If $\lambda_{1,2}$ are real $c^2 - 4mk > 0$ (corresponding to **overdamped** case).
2. If $\lambda_{1,2}$ are imaginary $c^2 - 4mk < 0$ (corresponding to **underdamped** case).
3. If $\lambda_1 = \lambda_2$ are real $c^2 - 4mk = 0$ (leading to $c_{cr} = \sqrt{4mk} = 2m\omega_n$) that corresponds to **critically damped** case (i.e., the system overshoots and comes back to rest).

Case 2 : Damped free vibration (1- DOF)

- Another approach to solve Newton's equation is the damping ratio (ζ). This is the ratio of the damping coefficient of the system to the critical damping coefficient:

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}} \rightarrow c = 2m\omega_n\zeta$$

$$\lambda_{1,2} = \omega_n[-\zeta \pm \sqrt{\zeta^2 - 1}]$$

- The three types of motions can then be defined by the damping ratio as:

  1. $\zeta > 1$ (for overdamped case);
  2. $\zeta < 1$ (for underdamped case) and
  3. $\zeta = 1$ for the critically damped case.

- The response of the system in terms of these two roots is defined as:

$$x(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$$

Case 2 : Damped free vibration (1- DOF)

For the underdamped case where the damping ratio range is $0 < \zeta < 1$ this leads to:

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_d j$$

$$\omega_d = \sqrt{1 - \zeta^2}$$

Assume initial values $x(0) = x_0$ and $\dot{x}(t) = v_0$

$$x(t) = Ae^{-\zeta\omega_n t} sin(\omega_d t + \phi)$$

$$x(t) = \sqrt{\frac{(v_0 + x_0\zeta\omega_n)^2 + (x_0\omega_d)^2}{\omega_d^2}} e^{-\zeta\omega_n t} sin\left(\omega_d t + tan^{-1}\left(\frac{x_0\omega_d}{v_0 + x_0\zeta\omega_n}\right)\right)$$

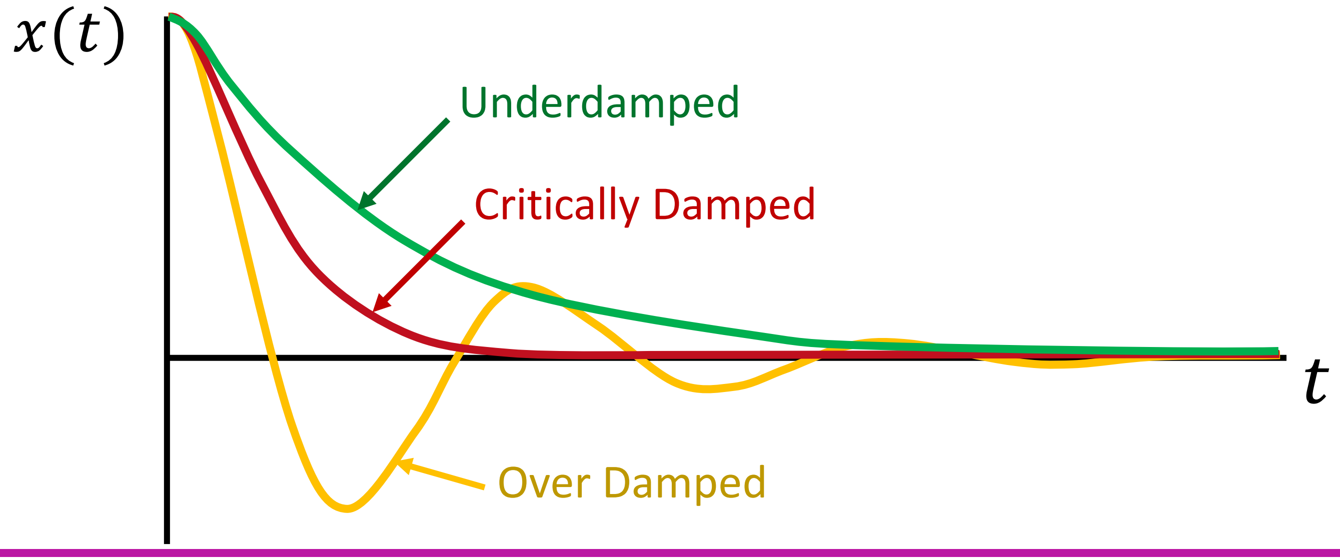
Case 2 : Damped free vibration (1 dof)

- If we follow the same procedure, the solution of overdamped case is given by:

$$x(t) = a_3 e^{(-\zeta\omega_n + \omega_d)t} + a_4 e^{(-\zeta\omega_n - \omega_d)t}$$

- Similarly, for the critically damped case, the solution is given by:

$$x(t) = [x_0 + (v_0 + \omega_n x_0)t]e^{-\omega_n t}$$

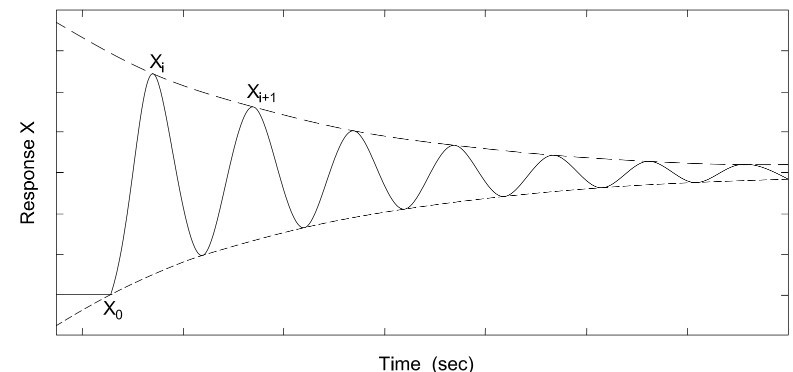


How can we practically assess damping ?

A practical way to assess damping that is broadly applicable in the area of ship hydrodynamics is the damping decay test. This can be mathematically expressed using the log decrement that is the natural logarithm of the ratio of two successive amplitudes.

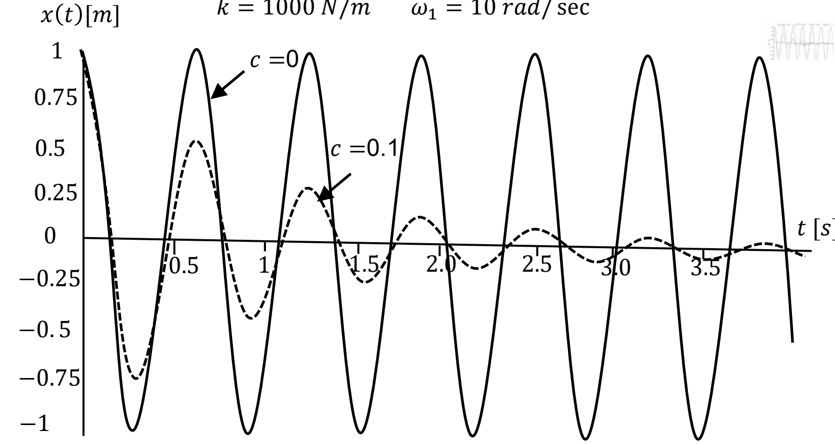
$$\delta = ln\frac{X_1}{X_2} = ln\frac{Ae^{-\zeta\omega_n t_1}}{Ae^{-\zeta\omega_n (t_1+T_d)}} = ln\, e^{\zeta\omega_n T_d} = \zeta\omega_n T_d$$

$$\because T_d = 2\pi/\omega_d \rightarrow \therefore \delta = \frac{2\pi\zeta\omega_n}{\omega_d} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$



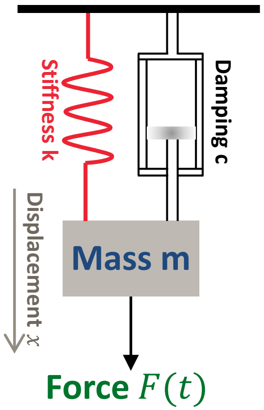
Since the damping ratio is very small in that case, the log decrement can be approximated by:

$$\delta = 2\pi\zeta$$



Case 3 : Forced Vibration – 1 DOF

Consider adding harmonic excitation to the vibration system where $F(t)$ varies in sinusoidal manner instead of being arbitrary function in time:



$$m\ddot{x} + c\dot{x} + kx = F(t) = F_0\, cos(\omega t)$$

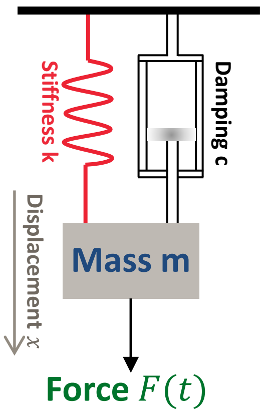
$$\rightarrow \ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = f_0 \cos(\omega t)$$

$$f_0 = F_0 / m$$

This is a differential equation of the 2nd order. Accordingly, it is prone to a general and particular solution which when combined together they may give the response function of the system.

Case 3 : Forced Vibration – 1 DOF

Consider adding harmonic excitation to the vibration system where $F(t)$ varies in sinusoidal manner instead of being arbitrary function in time:



$$m\ddot{x} + c\dot{x} + kx = F(t) = F_0\, cos(\ \omega t)$$

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = f_0\, cos(\omega t)$$

$$f_0 = F_0/m$$

This is a differential equation of the 2nd order. Accordingly, it is prone to a general and particular solution which when combined together they may give the response function of the system. The **general solution** is given when the left-hand side of the equation is equal to zero.

$$\ddot{x}_g(t) + 2\zeta\omega_n\dot{x}_g(t) + \omega_n^2 x_g(t) = 0$$

$$x_g(t) = Ae^{-\zeta\omega_n t}\, sin(\omega_d t) + \phi$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Case 3 : Forced Vibration – 1 DOF

- The **particular solution** is defined as:

$$\ddot{x}_p(t) + 2\zeta\omega_n\dot{x}_p(t) + \omega_n^2 x_p(t) = f_0\, cos(\,\omega t)$$

- There are two possible trial solutions to the particular solution namely

$$x_p(t) = A_s\, cos(\,\omega t) + B_s\, sin(\,\omega t) \qquad or \qquad x_p(t) = X\, cos(\,\omega t - \theta)$$

where $X^2 = A_s^2 + B_s^2, \qquad \theta = tan^{-1}(\,B_s/A_s)$

- Substituting the trial solution in the equation of motion leads to:

$$(-A_s\omega^2 + \ 2B_s\zeta\omega_n\omega + A_s\omega_n^2 - f_0) \cdot cos(\,\omega t) + (-B_s\omega^2 - 2A_s\zeta\omega_n\omega + Bs\omega_n^2) \cdot sin(\,\omega t) = 0$$

- For this equation to be zero at any time t, the two coefficients multiplied by $sin(\,\omega t)$ and $cos(\,\omega t)$ must be zero.

Case 3 : Forced Vibration – 1 DOF

- Solving these two equations we can find the two unknowns:

$$A_s = \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$B_s = \frac{2\zeta\omega_n\omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}$$

$$x_p(t) = \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} cos(\omega t) + \frac{2\zeta\omega_n\omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} sin(\omega t)$$

- The second particular solution after solving the unknowns X and θ becomes:

$$x_p(t) = X\, cos(\omega t - \theta)$$

$$x_p(t) = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}\, cos\left(\omega t - arctan\left[\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right]\right)$$

Case 3 : Forced Vibration – 1 DOF

- The full solution is the summation of the general solution $x_g(t)$and the particular solution $x_p(t)$

$$x(t) = x_g(t) + x_p(t)$$

$$x(t) = \ Ae^{-\zeta\omega_n t}\,sin(\,\omega_d t + \phi) + X\,cos(\,\omega t - \theta)$$

- A and ϕ can be obtained assuming initial conditions $x(0) = x_0$, and $\dot{x}(0) = v_0$

$$\phi = arctan[\frac{\omega_d(x_0 - X\,cos\,\theta)}{v_0 + (x_0 - X\,cos\,\theta)\zeta\omega_n - \omega X\,sin\,\theta}]$$

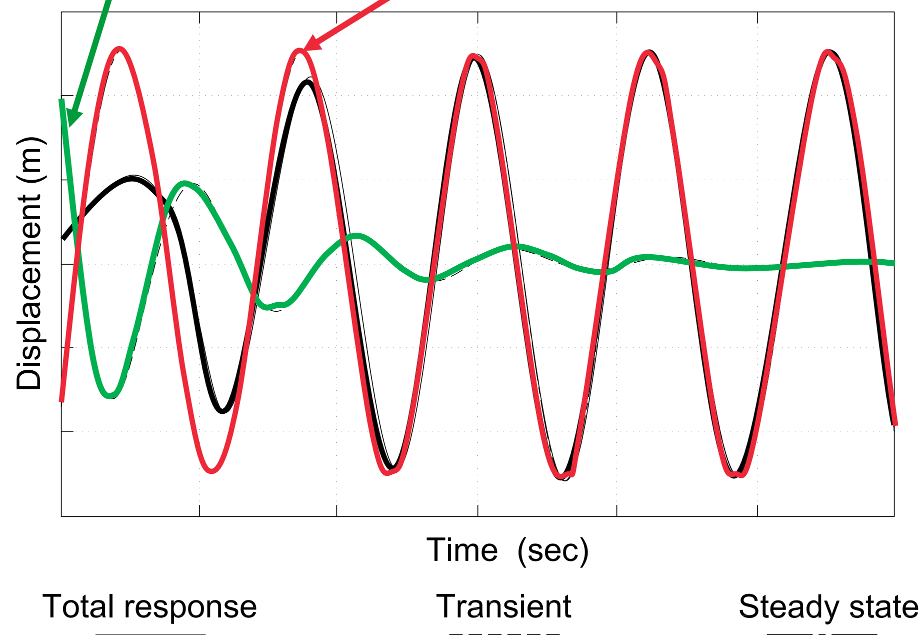
$$A = \frac{x_0 - X\,cos\,\theta}{sin\,\phi}$$

Case 3 : Forced Vibration – 1 DOF

$$x(t) = \boxed{Ae^{-\zeta\omega_n t}\, sin(\,\omega_d t + \phi)} + \boxed{X\, cos(\,\omega t - \theta)}$$

- The first term in the full solution is the **transient solution**, which tends to zero as the time goes to infinity.
- Second term is the **steady oscillatory solution**.
- Assuming steady state solution and neglecting the transient solution we got the full solution which equals the particular solution:

$$x_p(t) = X\, cos(\,\omega t - \theta)$$



Case 3 : Forced Vibration – 1 DOF

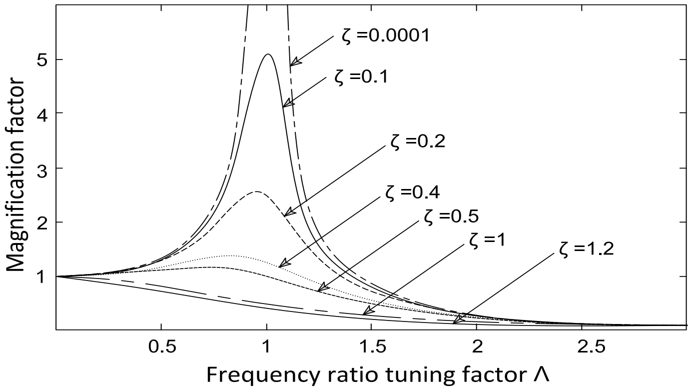
If we rewrite these equations as a function of the frequency ratio tuning factor $\Lambda = \dfrac{\omega_e}{\omega_n}$ we get the expression of

$$X = \frac{F_0}{k\sqrt{(1-\Lambda^2)^2 + (2\zeta\Lambda)^2}} cos(\omega_e t - \phi) \qquad \phi = tan^{-1}\left\{\frac{2\zeta\Lambda}{1-\Lambda^2}\right\}$$

The amplitude of the response can be represented in dimensionless form by the so-called magnification factor (Q)

$$Q = \frac{Amplitude\ of\ oscilation}{Equivalent\ static\ displacement} = \frac{X}{F_0/K} = \frac{1}{\sqrt{(1-\Lambda^2)^2 + (2\zeta\Lambda)^2}}$$

When the encounter frequency approaches the natural frequency, the **magnification factor** gets to extremely significant value, and such case is known by resonance.



Forced Vibrations due to Harmonic Excitation

- If we apply a Fourier integral on the excitation force of Newton's equation of motion external loading and response are defined as

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_P(\omega)e^{i\omega t}d\omega \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_x(\omega)e^{i\omega t}d\omega$$

- Then Newton's equation of motion becomes :

$$-m\int_{-\infty}^{\infty} A_x(\omega)\omega^2 e^{i\omega t}d\omega + c\int_{-\infty}^{\infty} A_x(\omega)i\omega e^{i\omega t}d\omega + k\int_{-\infty}^{\infty} A_x(\omega)e^{i\omega t}d\omega = \int_{-\infty}^{\infty} A_P(\omega)e^{i\omega t}d\omega$$

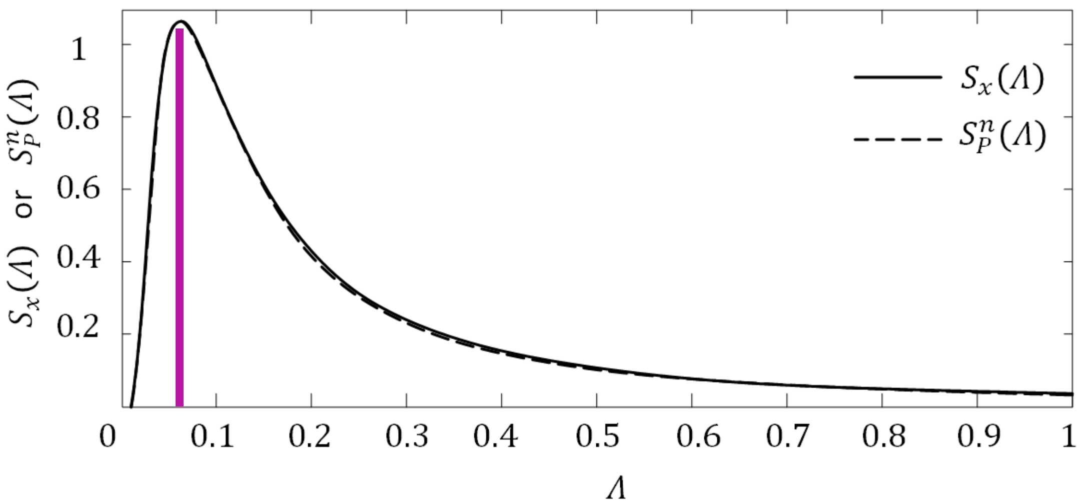
- For $A_x(\omega) = \dfrac{A_P(\omega)}{-m\omega^2 + ci\omega + k}$ ⇨ $A_x(\omega^*) = \dfrac{A_P^n(\omega^*)}{-m\omega^{*2} + \delta i\omega^* + 1}$

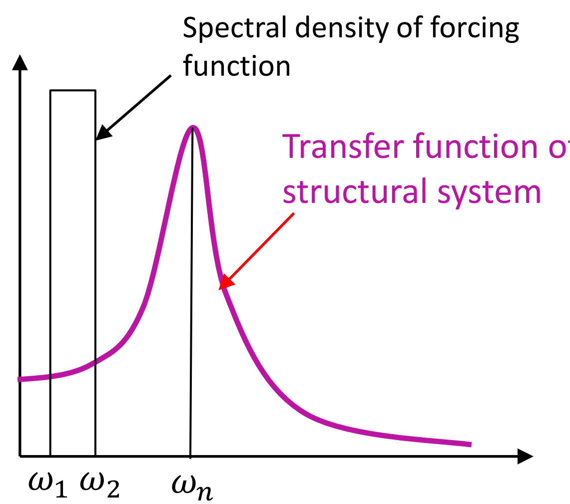
$$\begin{cases} A_P^n(\omega^*) = \dfrac{A_P(\omega)}{m\omega_n^2} \\ \delta = \dfrac{c}{m\omega_n} \end{cases}$$

Quasi-Static Response

- **At sub-critical case (also known as quasi-static response)** the system can reach high values of spectral density at small frequencies relative to the natural frequency and the stiffness has the major effect on the system
- Only stiffness affects the system response

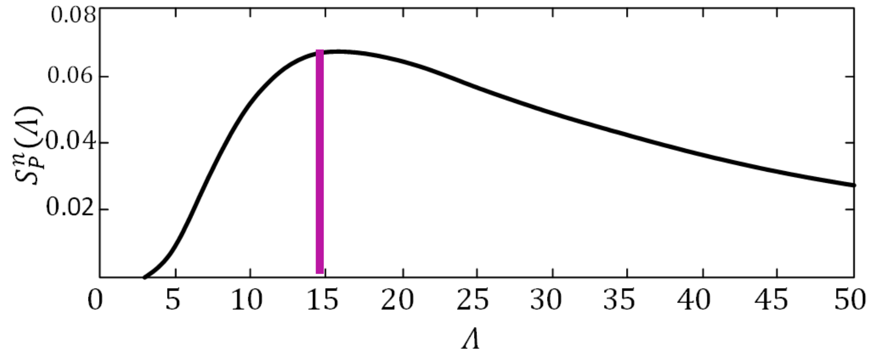
$$S_x(\Lambda) = \frac{S_P^n(\Lambda)}{(1-\Lambda^2)^2 + \delta^2\Lambda^2} \rightarrow S_x \approx S_P^n$$

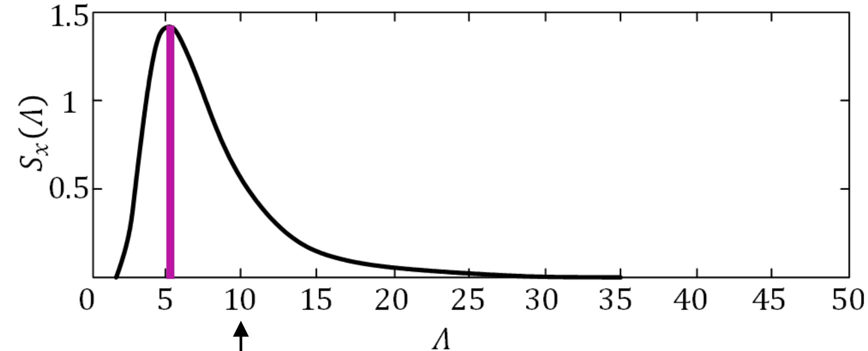




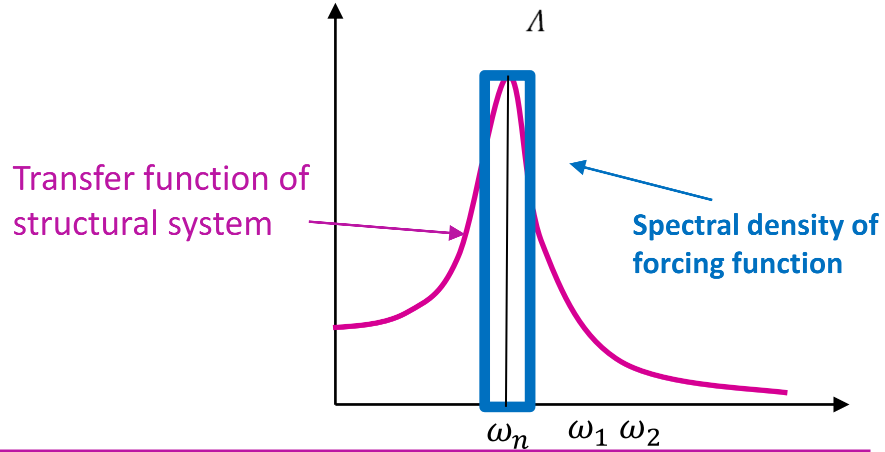
Dynamic Response

- At super-critical stage (also known as dynamic response) the highest values of spectral density lie in only high values of frequencies with respect to the natural frequency and damping plays an important role:
- **Only inertia forces affect the system response**





$$S_x(\Lambda) = \frac{S_P^n(\Lambda)}{(1-\Lambda^2)^2 + \delta^2\Lambda^2} \rightarrow S_x \approx \frac{S_P^n}{\Lambda^4}$$

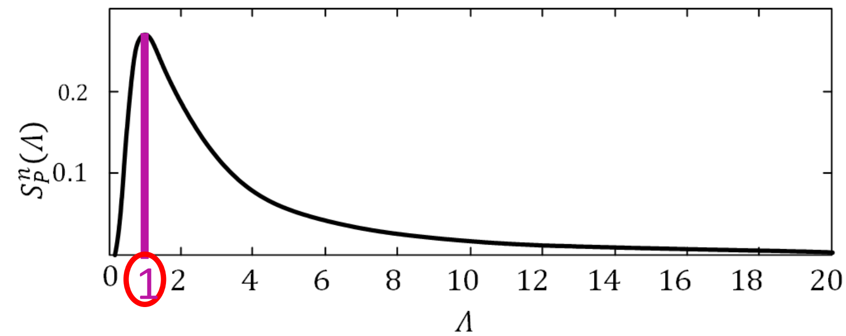


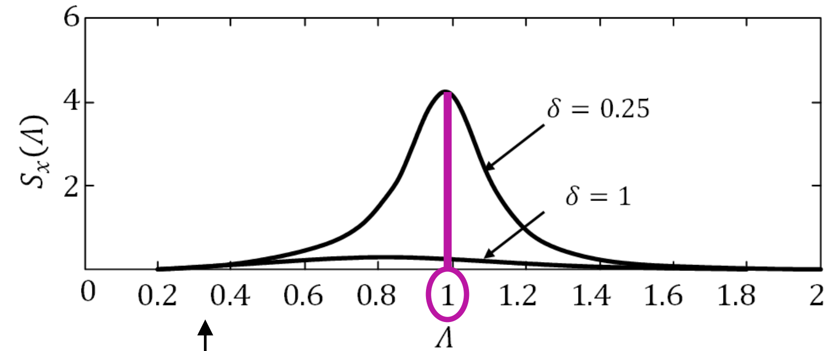
Resonance

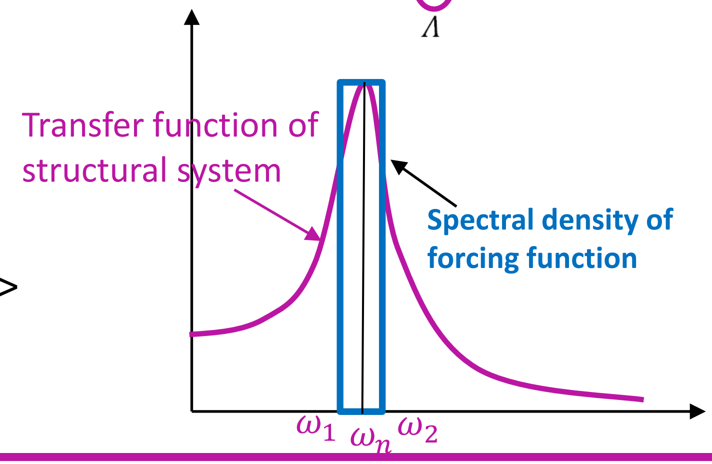
- At resonance condition when there is very low damping the frequency ratio ω^*approaches unity. The denominator approaches zero, and the spectral density approaches extremely large value:

- Serious problems which can be controlled only by adjusting damping

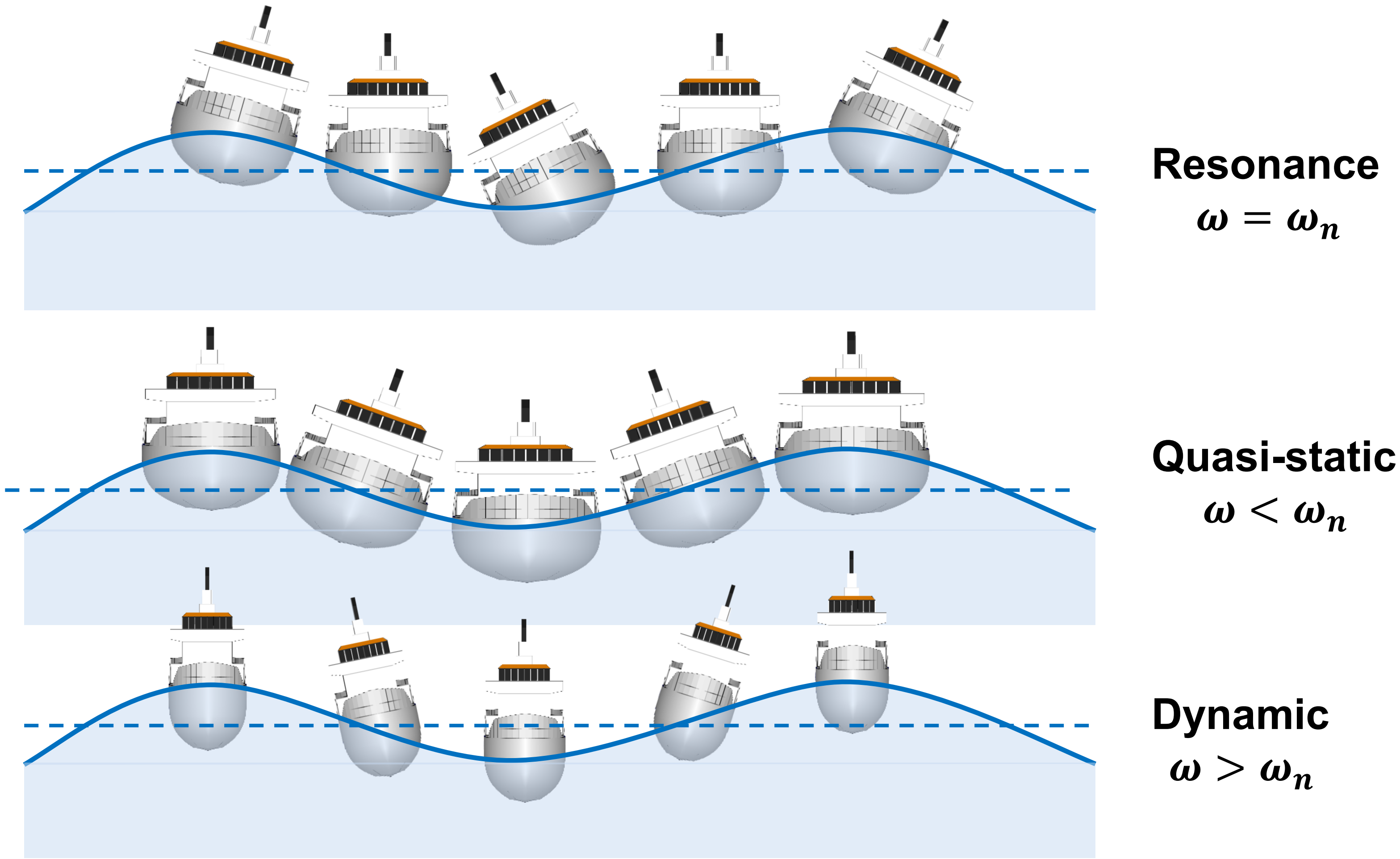
$$S_x(\Lambda) = \frac{S_P^n(\Lambda)}{(1-\Lambda^2\,)^2 + \delta^2\Lambda^2 \rightarrow \approx 0} \rightarrow S_x >>>>$$







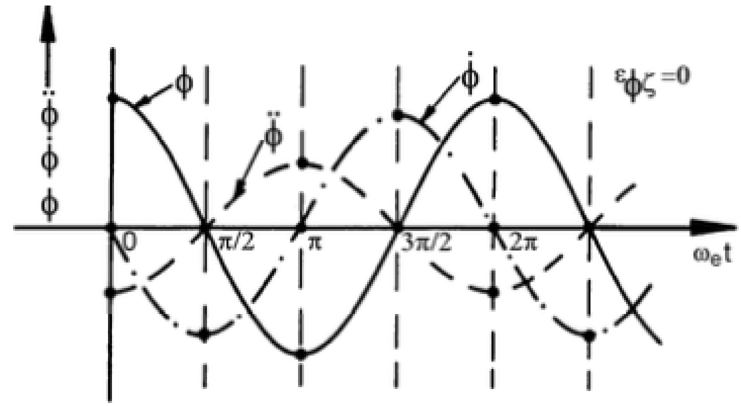
Dynamic behavior in different dynamic states



Preamble to Lecture 6 – Ship Eq. of Motion

As the functions of motion are trigonometric, there is relation between displacement, velocity and acceleration, i.e.

$$\phi = \phi_a \cos(\omega_e t + \varepsilon_{\phi\zeta})$$
$$\dot{\phi} = -\omega_e \phi_a \sin(\omega_e t + \varepsilon_{\phi\zeta}) = \omega_e \phi_a \cos(\omega_e t + \varepsilon_{\phi\zeta} + \pi/2)$$
$$\ddot{\phi} = -\omega_e^2 \phi_a \cos(\omega_e t + \varepsilon_{\phi\zeta}) = \omega_e^2 \phi_a \cos(\omega_e t + \varepsilon_{\phi\zeta} + \pi)$$



With these relations, the equation of motion for all 6 components is given as

$$[-\omega_e^2(M + A) + i\omega_e N + S]\hat{\vec{u}} = \hat{\vec{F}}_e$$

$$M = \begin{bmatrix} m & 0 & 0 & 0 & mz_g & 0 \\ 0 & m & 0 & -mz_g & 0 & mx_g \\ 0 & 0 & m & 0 & -mx_g & 0 \\ 0 & -mz_g & 0 & \theta_{xx} & 0 & -\theta_{xz} \\ mz_g & 0 & -mx_g & 0 & \theta_{yy} & 0 \\ 0 & mx_g & 0 & -\theta_{xz} & 0 & -\theta_{zz} \end{bmatrix} \quad S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho g A_w & 0 & -\rho g A_w x_w & 0 \\ 0 & 0 & 0 & gm\overline{GM} & 0 & 0 \\ 0 & 0 & -\rho g A_w x_w & 0 & gm\overline{GM}_L & 0 \\ 0 & 0 & 0 & 0 & 0 & \theta_{zz}\omega_g^2 \end{bmatrix}$$

$$\theta_{xx} = \int (y^2 + z^2)\,\mathrm{d}m; \quad \theta_{xz} = \int xz\,\mathrm{d}m; \quad \text{etc.}$$

Summary and next steps

- Ship is a 6 DOF rigid body system
  - ✓ *3 translations surge, heave and sway*
  - ✓ *3 rotations pitch, yaw and roll.*

- Ship motions are influenced by
  - ✓ *The ship main particulars*
  - ✓ *Shape*
  - ✓ *Weights*
  - ✓ *Excitation loads*

- The most significant motions are those that have a restoring force associated with them : *Pitch, Heave and Roll*

- In motions analysis we usually use the encounter frequency (ω_e) to account for the effect of ship forward speed.

- The derivation of ship motions equations are reflected by rigid body dynmic system theory - *Free undamped or damped vibration subject to harmonic excitation or random loading*

Thank you !