Lecture 5 Introduction to ship motions

1. Introduction

Ship seakeeping is a term that reflects the ability of a vessel to withstand rough conditions at sea. It therefore involves the study of the motions of a ship when subjected to waves, and the resulting effects on humans, systems and mission capability (Lloyd, 1989). With fast computers and sophisticated software readily available to designers, it is now possible for a vessel's seakeeping characteristics to be addressed much earlier in the design spiral. As shown in Figure 5-1 there are three main components that influence ship seakeeping responses namely**,** (a) the waves as input to the system, (b) ship system characteristics and (c) ship motions.

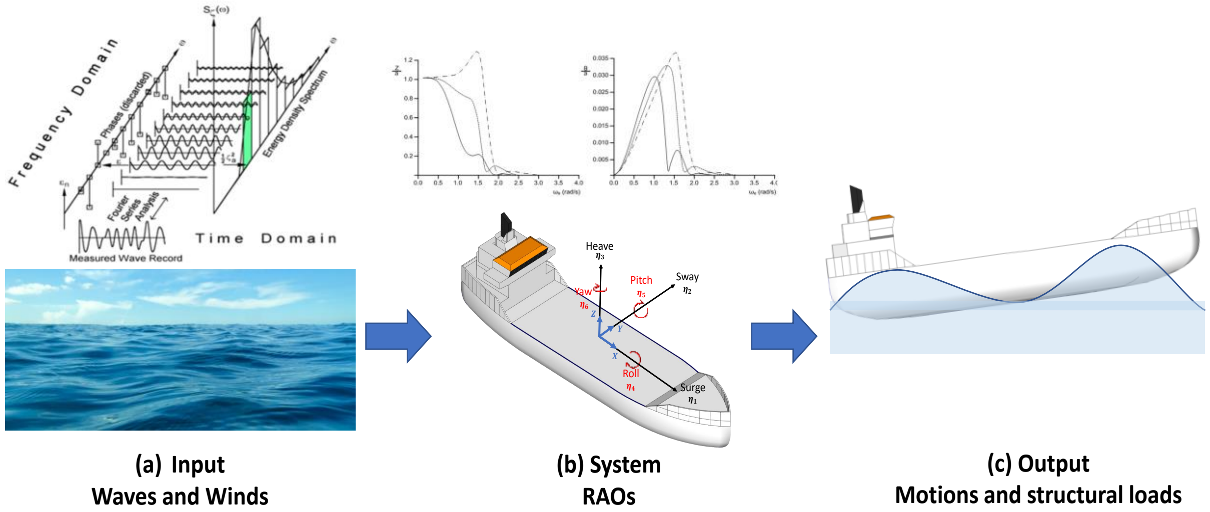

Figure 5-1 The principle of ship response in wind and waves (a) represents the ocean/enviromental conditions; (b) ship characteristics (c) ship motions and sea loads.

A vessel's general particulars (e.g., length, beam, draft), hull form and metacentric height influence seakeeping responses and in turn ship safety and performance (Hirdaris, 2021). For example, small length ships with classic hull forms possibly including bow flare while progressing at medium to high forward speeds suffer from large motions. On the other hand, long and bulkier ships experience lower motion amplitudes. Shallow drafts may lead to higher risks of keel emergence and bow slamming loads in rough seas. In turn, motions may also vary due to loading conditions and operational factors. Small hull form adjustments (e.g., reduction of the radius of curvature in way of the bilges) can marginally influence ship motions. Notwithstanding this, a large forward waterplane can reduce overall motions and the probability of keel emergence. Changes to overall ship proportions (e.g., beam to length or beam to draft ratios) may have important consequences. For example, reducing the draft of a ship (for a given length and beam) may reduce ship motion amplitudes. The ship beam relates with metacentric height. Whereas a large metacentric height improves initial stability, it may also lead to high hull natural frequencies which are usually associated with poor motion sickness indices. On the other hand, if the metacentric height is too small motions are smoother but the risk of capsize increases dramatically.

2. Basic definitions

A ship is a six degree of freedom (6-DoF) rigid body system. Motions (1 - 3) are linear displacements (translations) known as surge, sway and heave. Motions (4 - 6) are rotations known as roll, pitch and yaw. All motions are measured relative to the ship as shown in Figure 5-2.

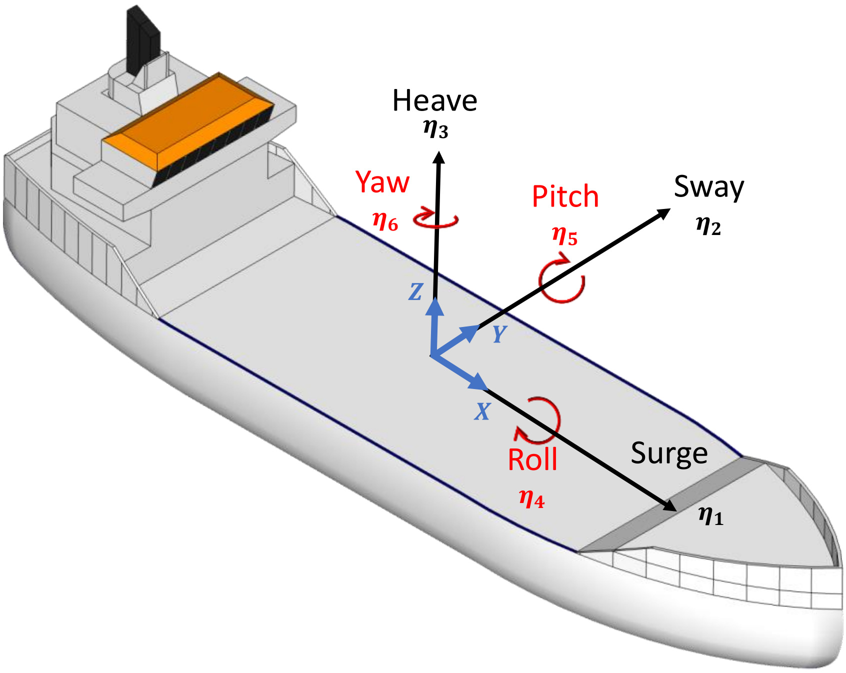

Figure 5-2 Ship seakeeping degrees of freedom

- **Surge** describes the horizontal oscillations of the ship toward the bow and the stern.
- **Sway** is a side - side motion. A vessel moving to starboard travels in positive sway direction.
- **Heave** is the vertical motion. By convention, positive heave points downwards (toward the water bottom). So, a vessel that is sinking into the water (i.e. increasing her draft) is moving in the positive heave direction.
- **Roll** is a rotational motion about the surge axis. If the starboard and port sides move vertically but in opposite directions (i.e. the starboard side is moving up while the port side is moving down). By convention positive roll angles correspond to the starboard moving downwards while the port side moves in the opposite direction.
- **Pitch** is the rotational motion about the sway axis. When pitching, the bow and stem are moving vertically in opposite directions (i.e. when the bow is moving up and the stem is moving down). Pitch is positive when the bow is pointing upwards in relation to a level ship.
- **Yaw** is the rotational motion about the heave axis. It describes the turning motion of the ship. When the bow moves in the starboard direction, we assume that the yaw angle is positive.

Amongst the above mentioned 6 - DoF the most significant ones are those that have a restoring force associated with them. For example, a wave push to the vessel's side (known as the sway motion effect) may be inconvenient in terms of navigation. However, the effect is limited in time as there are no restoring forces. On the other hand, if a ship is pushed over so that her starboard deck drops while waves pass over, returning to her original upright position is critical in terms of avoiding capsize.

Figure 5-3 illustrates an example of restoring forces emerging from heave movements. In linear seakeeping we can assume that heave is a rigid body response proportional to the distance displaced. This is because of the disparity between displacement and buoyancy forces that may be considered linear for different waterlines. Of course, ships that have a large water plane area for their displacement will experience much greater heave restoring forces than ships with small water plane areas. So "*beamy*" ships such as tugs and fishing vessels will suffer short period heave oscillations and high heave accelerations. Conversely, ships with small water plane areas will have much longer heave periods and experience lower heave accelerations. In general, as acceleration reduces, comfort is reassured. This concept is taken to extremes in the case of offshore floating platforms that have very small water plane area compared to their displacement.

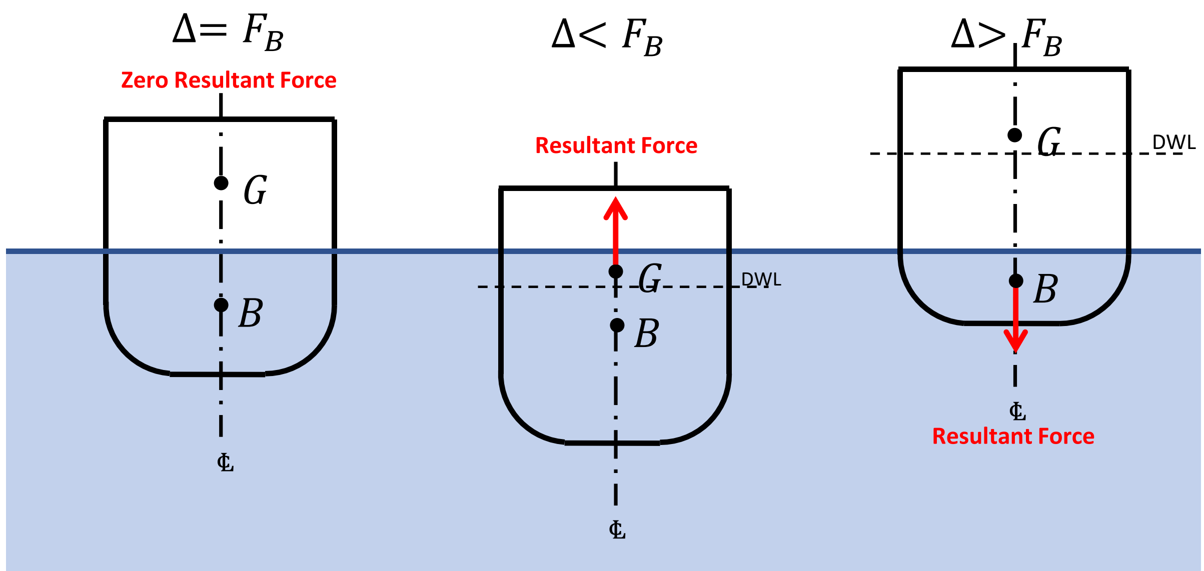

Figure 5-3 Simplistic idealization of the heave restoring force (F_B = Buoyancy; Δ = displacement)

When ship dynamics are accounted for, the encounter frequency (ω_e) with the waves is used instead of the absolute wave frequency (ω). This is because a ship that is moving relative to the waves, will meet successive peaks and troughs in short or long-time intervals. Her dynamic behavior depends on whether she is advancing into the waves or travelling in their direction. If we assume that the waves and the ship are on a straight course, the frequency with which the ship will encounter a wave crest depends on the wavelength (λ) defined as the distance between the wave crests, the speed (or celerity) of the waves (c), the speed of the ship (U) and the relative angle between the ship heading with the wave heading (μ), see Figure 5-4. This is the reason why the encounter period is defined as the distance traveled (λ) divided by the speed the ship encounters the waves ($c - U\,cos(\mu)$). Therefore, the encounter frequency is defined as

$$\omega_e = \frac{2\pi}{T_E} = \frac{2\pi}{\lambda}(c - Ucos(\mu)) = k\left(c - Ucos(\mu)\right) = \omega - kUcos\mu \qquad (5\text{-}1)$$

In deep waters the wave number $k = \frac{\omega^2}{g}$ leading to,

$$\omega_e = \omega - \frac{\omega^2}{g}Ucos(\mu) \qquad (5\text{-}2)$$

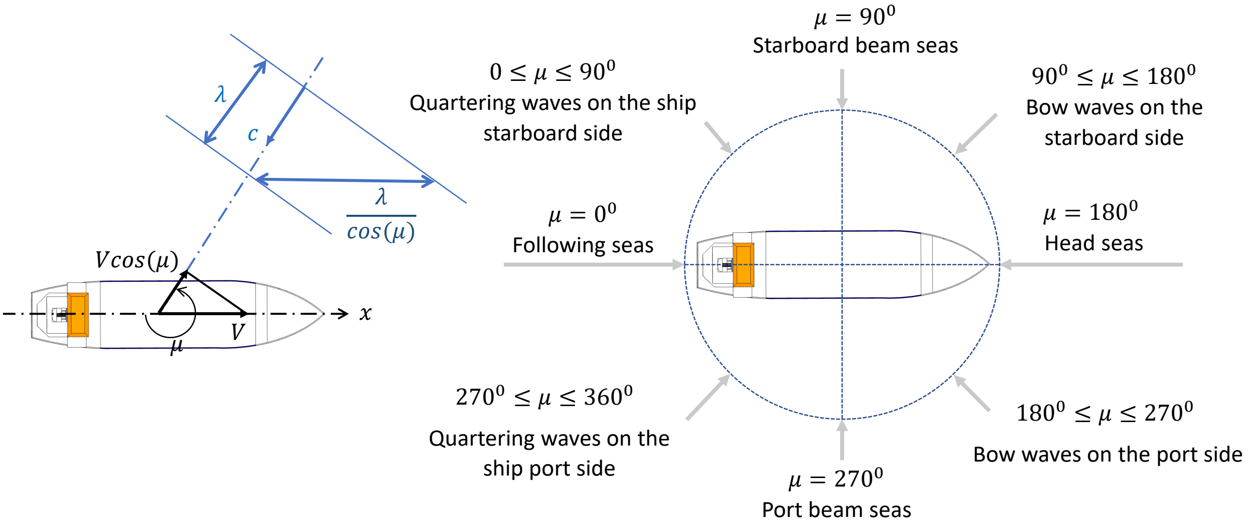

Figure 5-4 Ship seakeeping encounters idealizations

To describe the position and orientation of a ship, different coordinate systems may be used (see Figure 5-5). The **earth fixed inertial coordinate system** $\{n\}$ is used to define the position of the vessel on the earth, the direction of wind, waves and current. It is determined by a tangent plane attached at a point of interest (O_n). The positive unit vector (n_1) points towards the true North, (n_2) points towards the East, and (n_3) points towards the interior of the earth. When using such system, the inertial assumption is considered reasonable because the velocity of marine vehicles is relatively small and thrust forces due to the rotation of the Earth may be considered negligible relatively to the hydrodynamic forces. The **body-fixed coordinate system** (O_b) is fixed to the hull and is used to express velocity and acceleration measurements taken onboard or for the idealization of performance motion indices. The positive unit vector (b_1) points towards the bow, (b_2) points towards starboard and (b_3) points downwards. For ships, the axes of this frame are often chosen to coincide with the principal axes of inertia. The **seakeeping coordinate system** (s) located at the center of gravity of the vessel moves at the average speed of the vessel and follows her path. It is used to define the wave elevation at the vessel's average location and to compute the hydrodynamic forces using software. This system is fixed to the vessel's equilibrium state, which is defined by the average speed and heading. The positive unit vector (s_1) points forward and is aligned with the average forward speed. The positive unit vector (s_2) points towards starboard, and (s_3) points downwards.

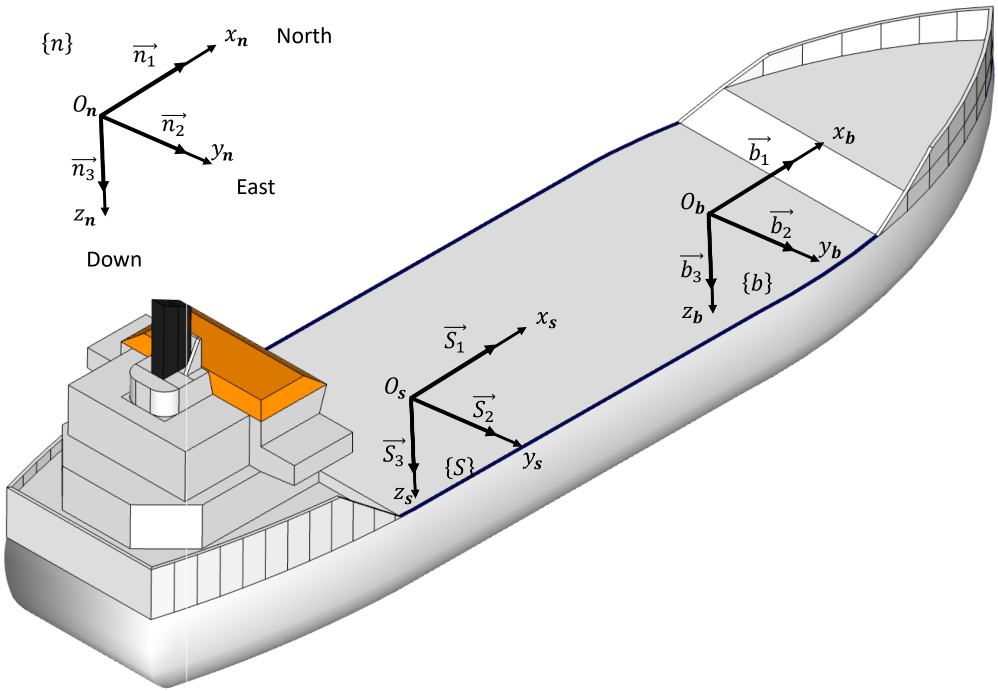

Figure 5-5 Illustration of the three reference frames

3. The dynamics of the rigid ship

The fundamental principles discussed in this section are discussed in various basic textbooks dealing with structural dynamics and stochasticity (e.g. Brouwers, 2006; Newland, 2012). However, for practical reasons the discussion is presented in an analogous format to principles of naval architecture dynamics and seakeeping in rough weather (e.g., see Lloyd, 1989 ; Naess and Moan, 2013).

3.1 The spring mass system analogy

The seakeeping behavior of a ship is similar to the classic oscillatory response of a damped spring-mass system. If we consider the general form of the typical single degree of freedom (1-DOF) system of such kind with force excitation varying in time while the mass is displaced in either direction, the spring will be compressed or placed in tension as shown in Figure 5-6). This will generate a *"restoring force"* that attempts to return the ship to her original location. Provided that the spring remains within its linear operating region, the size of the force will be proportional to the amount of displacement. However, because of inertia effects, the mass will overshoot from its original point of reference; hence the spring oscillations shall generate another linear restoring force in the opposite direction that enacts to restore the mass to its central position. This dynamic behavior will be repeated until the effects of the damper dissipate the energy stored by the system oscillations.

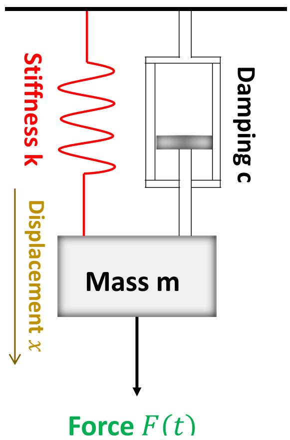

Figure 5-6 Typical spring-mass system with damper

For such system idealization Newton's 2nd law of motion applies as follows

$$\sum \vec{F} = m\ddot{x} \tag{5-3}$$

where $\sum \vec{F}$ is the total force, m is the mass of the body and $\ddot{x}$ is the acceleration. Decomposition of Eq (5-2) leads to

$$\begin{gathered} -kx - c\dot{x} + F = m\ddot{x} \Rightarrow \\ \Rightarrow m\ddot{x} + c\dot{x} + kx = F(t) \end{gathered} \tag{5-4}$$

where c is the damping coefficient and k is the stiffness.

From a physical viewpoint what is presented in Eq. (5-3) is similar to the case of a ship floating on waves as an 1-DOF system. The stiffness term is mainly attributed to buoyancy force. To realize the significance of this term just imagine the ideal case for which a ship undergoes pure heave motion. If you push the ship downwards in the water, based on *"Archimedes Principle"* (Hirdaris, 2021) there will be an extra buoyant force acting upwards in excess of the ship's displacement. If you then release the downward force on the ship, she will move up. Likewise, lifting a ship out of the water will result in lower buoyancy force than the ship's displacement. So, when released the ship will move down.

3.2 Free undamped vibration of 1- DOF system

If we assume the ship is a conservative system (i.e., no energy losses occur), Equation (5-4) becomes

$$m\ddot{x} + kx = 0 \tag{5-5}$$

Rigid body dynamic response can be extracted by assuming a sinusoidal solution $x = e^{\lambda t}$leading to

$$\lambda^2 m + k = 0,\ \lambda = \pm\sqrt{\frac{k}{m}} = \pm j\omega_n \tag{5-6}$$

where $\omega_n = \sqrt{k/m}$ represents that natural frequency of the oscillation. The response may then be defined as

$$\begin{gathered} x = A_1 e^{j\omega_n t} + B_1 e^{-j\omega_n t} = \\ = A\,sin(\omega_n t) + B\,cos(\omega_n t) = \\ = X\,sin(\omega_n t + \phi) \end{gathered} \tag{5-7}$$

In Eq (5-6) the amplitude $X = \sqrt{A^2 + B^2}$ and the phase $\phi = tan^{-1}(B/A)$.

If at the start of the oscillation (i.e., at t = 0) the ship displacement is x_0 the velocity becomes $\dot{x}(t = 0) = v_0$, leading to:

$$X = \frac{\sqrt{\omega_n^2 x_0^2 + v_0^2}}{\omega_n} \tag{5-8}$$

$$\phi = tan^{-1}\left(\frac{\omega_n x_0}{v_0}\right) \tag{5-9}$$

Thus, the final solution of the system displacement, Figure 5-7, velocity and acceleration become

$$\begin{aligned} x(t) &= X\sin(\omega_n t + \phi) = \frac{1}{\omega_n}\sqrt{\omega_n^2 x_0^2 + v_0^2}\,\sin(\omega_n t + \phi) \\ \dot{x}(t) &= X\omega_n \cos(\omega_n t + \phi) = \sqrt{\omega_n^2 x_0^2 + v_0^2}\,\cos(\omega_n t + \phi) \\ \ddot{x}(t) &= -X\omega_n^2 \sin(\omega_n t + \phi) = -\omega_n\sqrt{\omega_n^2 x_0^2 + v_0^2}\,\sin(\omega_n t + \phi) \end{aligned} \tag{5-10}$$

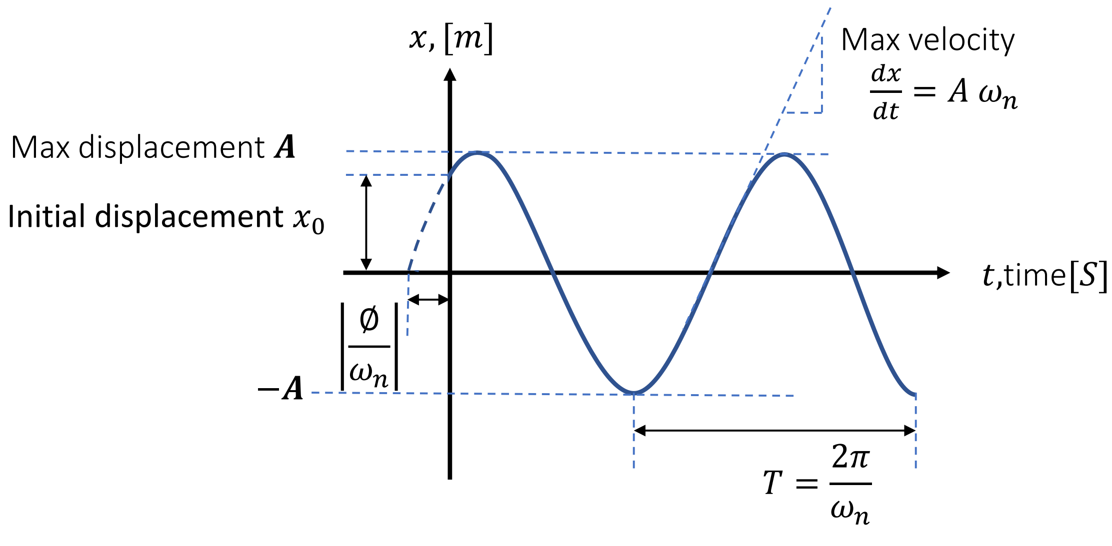

Figure 5-7 Free undamped vibration response

3.3 Free damped vibration of single DOF system

In reality, the ship will not behave as a conservative system; i.e. the amplitude of oscillation will reduce with time due to damping effects. Even a low level of damping will allow for several oscillations before she comes to rest. Thus, if we may still assume free oscillations and accordingly Eq. (5-7) takes the form :

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{5-11}$$

Assuming sinusoidal solution $x = e^{\lambda t}$ the equation becomes

$$m\lambda^2 + c\lambda + k = 0, \quad \lambda_{1,2} = \frac{-c + \sqrt{c^2 - 4mk}}{2m} \tag{5-12}$$

There are three different types of motions associated to the above namely:

- if $\lambda_{1,2} \in \mathfrak{R}$ then the determinant $c^2 - 4mk > 0$ and the system is considered **overdamped** ; i.e. the response is very slow and looks like an exponential decay signal.
- if $\lambda_{1,2} \in \mathfrak{Im}$ the determinant $c^2 - 4mk < 0$ and the system is **underdamped**; i.e. the response is very fast and looks like a rapidly decaying oscillation where the amplitudes of oscillation look smaller and smaller until equilibrium is reached.
- if $\lambda_1 = \lambda_2 \in \mathfrak{R}$ and $c^2 - 4mk = 0$ then the damping factor becomes critical, $c_{cr} = \sqrt{4mk} = 2m\omega_n$. In this case the system is **critically damped**; i.e. the system is allowed to overshoot and then come back to equilibrium state (i.e. at rest) relatively fast and without any oscillations.

A dimensionless system parameter that describes how rapidly the oscillations decay is the damping ratio (ζ) defined as

$$\zeta = \frac{c}{c_{cr}} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}} \rightarrow c = 2m\omega_n\zeta \tag{5-13}$$

where *c* is the damping coefficient, c_{cr} is the critical damping coefficient. If we use this dimensionless notation the roots of Equation (5-12) can be expressed as:

$$\lambda_{1,2} = \omega_n[-\zeta \pm \sqrt{\zeta^2 - 1}] \tag{5-14}$$

Accordingly, the three types of motions can be defined as:

- $\zeta > 1$ for the overdamped case
- $0 < \zeta < 1$ for the underdamped case and
- $\zeta = 1$ for the critically damped case

The linear sinusoidal response of the system in terms of the roots expressed in Equation (5-14) can be defined as:

$$x(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t} \tag{5-15}$$

Therefore, λ_1 and λ_2 are part of the solution, for an underdamped case this leads to:

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\zeta\omega_n \pm \omega_d j \tag{5-16}$$

where

$$\omega_d = \sqrt{1 - \zeta^2} \tag{5-17}$$

If we follow similar process to the one demonstrated in Section 3.2, we can obtain the two unknowns A and ϕ in equation (5-18). Hence, the response becomes:

$$x(t) = Ae^{-\zeta\omega_n t}\, sin(\,\omega_d t + \phi)$$

for

$$x(t) = \sqrt{\frac{(v_0 + x_0\zeta\omega_n)^2 + (x_0\omega_d)^2}{\omega_d^2}}\, e^{-\zeta\omega_n t} \tag{5-18}$$

$$\text{and } sin\left(\omega_d t + tan^{-1}\left(\frac{x_0\omega_d}{v_0 + x_0\zeta\omega_n}\right)\right)$$

So the response for overdamped and critically damped cases are given by Eqs. (5-19) and (5-20) respectively

$$x(t) = a_3 e^{(-\zeta\omega_n + \omega_d)t} + a_4 e^{(-\zeta\omega_n - \omega_d)t} \tag{5-19}$$

$$x(t) = [x_0 + (v_0 + \omega_n x_0)t]e^{-\omega_n t} \tag{5-20}$$

Noted that the response of the overdamped solution is not oscillatory, which is considered a preferable case however it is difficult to achieve. Underdamped response is also non-oscillatory, but it provides the fastest solution that return to zero after time.

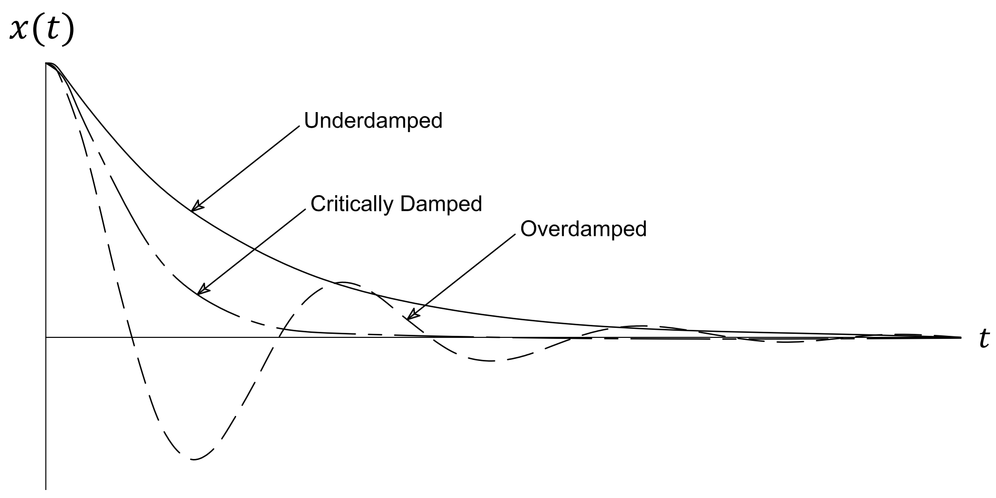

Figure 5-8 The three cases of damped free vibration response

A practical way to assess damping that is broadly applicable in ship dynamics is the damping decay Figure 5-9. This can be mathematically expressed using the log decrement δ that is the natural logarithm of the ratio of two successive amplitudes. The natural logarithm of the ratio of the first two successive amplitudes $X_1$and $X_2$is defined based on the underdamped solution as follows

$$\delta = ln\frac{X_1}{X_2} = ln\frac{Ae^{-\zeta\omega_n t_1}}{Ae^{-\zeta\omega_n(t_1+T_d)}} = ln\, e^{\zeta\omega_n T_d} = \zeta\omega_n T_d \tag{5-21}$$

$$T_d = \frac{2\pi}{\omega_d} \text{ and therefore } \delta = \frac{2\pi\zeta\omega_n}{\omega_d} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{5-22}$$

Since the damping ratio is very small in that case, the log decrement can be approximated by

$$\delta = 2\pi\zeta \tag{5-23}$$

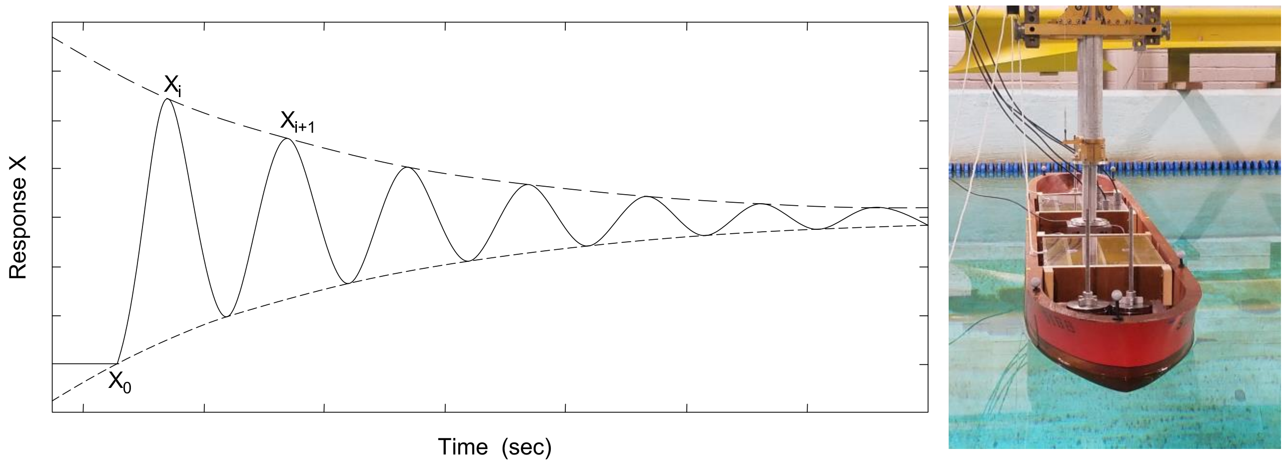

Figure 5-9 Roll angle decay response (left) of a tanker ship model (right) ©Jane-Frances 2020

3.4 Forced vibration of 1- DOF system

In practice ships never operate in conditions that there is no heaving, rolling or pitching. Therefore, to suitably idealize ship oscillations in time, it is necessary to account for the energy from waves. This energy is required to overcome the energy dissipated because of damping. In practical terms, fluid forces from the wave environment representing *"the injected energy"* should be accounted for when evaluating the ship motions that depend on the mass of the ship. To maintain ship oscillations, a force having the same frequency as the *"simple harmonic motion"* of the system is required. To illustrate the above principle let us consider adding a harmonic excitation to the vibration system where $F(t)$ varies in sinusoidal manner instead of being arbitrary function in time. In this case Eq. (5-7) becomes

$$m\ddot{x} + c\dot{x} + kx = F(t) = F_0\, cos(\,\omega_e t) \Rightarrow \tag{5-24}$$

where F_0 is the forcing amplitude and ω_e the encounter frequency representing the frequency at which the waves past the ship. The solution to this equation will be a system that experiences transient dynamics to the point that the ship's natural buoyant / damping response to the initial displacement and then an equilibrium solution will have the same frequency as the excitation force. Implementation of the same process followed in section 3.4 leads to the expression

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = f_0\, cos(\omega t) \text{ for } f_0 = F_0/m \tag{5-25}$$

The general solution of the 2nd order differential Equation (5-25) is given when

$$\ddot{x}_g(t) + 2\zeta\omega_n\dot{x}_g(t) + \omega_n^2 x_g(t) = 0 \tag{5-26}$$

leading to,

$$x_g(t) = Ae^{-\zeta\omega_n t}\, sin(\omega_d t + \phi), \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} \tag{5-27}$$

where terms A and ϕ represent the amplitude and phase of the response. The particular solution of Equation (5-25) is given by solving the differential equation

$$\ddot{x}_p(t) + 2\zeta\omega_n\dot{x}_p(t) + \omega_n^2 x_p(t) = f_0\, cos(\omega t) \tag{5-28}$$

There are two possible trial solutions namely,

$$x_p(t) = A_s\, cos(\omega t) + B_s\, sin(\omega t) \text{ or } \qquad x_p(t) = X\, cos(\omega t - \theta) \tag{5-29}$$

where

$$X^2 = A_s^2 + B_s^2, \qquad \theta = tan^{-1}(B_s/A_s) \tag{5-30}$$

If we substitute the trial solution in the equation of motion we get,

$$(-A_s\omega^2 + 2B_s\zeta\omega_n\omega + A_s\omega_n^2 - f_0)\cdot cos(\omega t) + (-B_s\omega^2 - 2A_s\zeta\omega_n\omega + Bs\omega_n^2)\cdot sin(\omega t) = 0 \tag{5-31}$$

For this equation to be zero at any time t, the two coefficients multiplied by $sin(\omega t)$ and $cos(\omega t)$ must be zero. Thus

$$-A_s\omega^2 + 2B_s\zeta\omega_n\omega + A_s\omega_n^2 - f_0 = 0 \tag{5-32}$$

$$-B_s\omega^2 - 2A_s\zeta\omega_n\omega + Bs\omega_n^2 = 0 \tag{5-33}$$

Solving these two equations we can find the two unknowns

$$A_s = \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \quad \text{and}$$

$$B_s = \frac{2\zeta\omega_n\omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} \tag{5-34}$$

The particular solution after solving the unknowns becomes,

$$x_p(t) = A_s\, cos(\omega t) + B_s\, sin(\omega t)$$

$$x_p(t) = \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} cos(\omega t) + \frac{2\zeta\omega_n\omega f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2} sin(\omega t) \tag{5-35}$$

or

$$x_p(t) = X\, cos(\omega t - \theta)$$

$$\text{where } X = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}} \quad \text{and} \quad \theta = tan^{-1}\left[\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right]$$

$$x_p(t) = \frac{f_0}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n\omega)^2}}\, cos\left(\omega t - arctan\left[\frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2}\right]\right) \tag{5-36}$$

Eventually, the full solution is the summation of the general solution and the particular solution

$$x(t) = x_g(t) + x_p(t)$$

$$x(t) = Ae^{-\zeta\omega_n t} sin(\omega_d t + \phi) + X\, cos(\omega t - \theta) \tag{5-37}$$

If we solve A and ϕ using the initial conditions $x(0) = x_0$, and $\dot{x}(0) = v_0$

$$\phi = arctan\left[\frac{\omega_d(x_0 - X\, cos\,\theta)}{v_0 + (x_0 - X\, cos\,\theta)\zeta\omega_n - \omega X\, sin\,\theta}\right] \tag{5-38}$$

$$A = \frac{x_0 - X\, cos\,\theta}{sin\,\phi} \tag{5-39}$$

The first term in the full solution is the transient solution, which tends to zero as the time goes to infinity, while the second term is the steady oscillatory solution (see Figure 5-10). The second term is of more importance as it is the steady solution. In many cases, we neglect the transient solution.

The full solution then reduces to the particular solution presented in Eq. (5-36). $x_p(t) = X\,cos(\omega t - \theta)$.

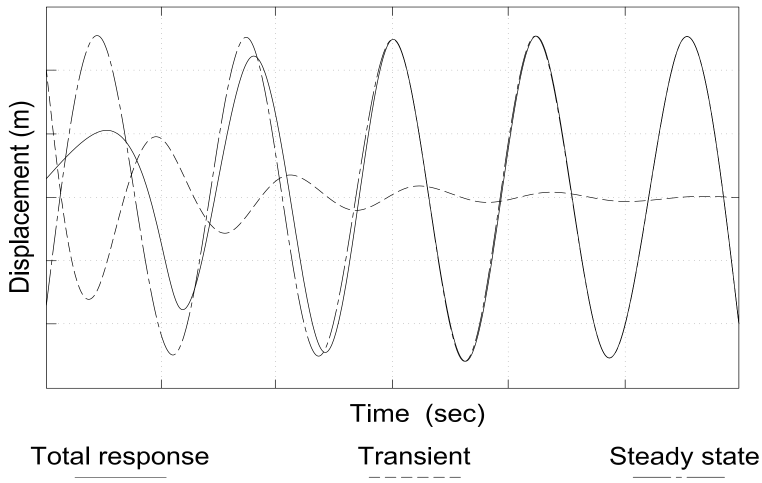

Figure 5-10 Harmonic excitation damped vibration

4. The concept of dynamic magnification factor

As explained in section 3.4 the equation of motion subject to a sinusoidal force is

$$m\ddot{x} + c\dot{x} + kx = F(t) = F_0\,cos(\,\omega_e t) \tag{5-40}$$

where F_0 is the forcing amplitude and ω_e the encounter frequency representing the frequency at which the waves past the ship. The solution to this equation expresses the dynamics of a system that experiences transient excitations to the point that the natural buoyant / damping response to the initial displacement and then an equilibrium solution will have the same frequency as the excitation force. A trial solution of the order $x = x_0 cos(\omega_e t - \phi)$, leads to $\dot{x} = -\omega_e x_0 sin(\omega_e t - \phi)$ and $\ddot{x} = -\omega_e^2 x_0 cos(\omega_e t - \phi)$. Thus, the solution to Eq. (5-36) becomes

$$X_0 = \frac{F_0}{\sqrt{(k - m\omega_e^2)^2 + c^2\omega_n^2}} \tag{5-41}$$

$$\phi = tan^{-1}\left(\frac{c\omega_n}{k - m\omega_e^2}\right) \tag{5-42}$$

In practice,

- the natural frequency defined as $\omega_n = \sqrt{\frac{k}{m}}$ expresses the frequency at which the system of stiffness (*k*) and mass (*m*) oscillates on its own when disturbed from equilibrium;
- the frequency ratio tuning factor $(\Lambda = \frac{\omega_e}{\omega_n})$ can be defined as the aspect ratio of the excitation to the natural frequency of the system;

- the damping ratio defined as $\zeta = \frac{c}{2\sqrt{km}}$ expresses the lost energy encompassed in same system with damping factor c;

Therefore Eqs.(5-41) and (5-42) take the form

$$X_0 = \frac{F_0}{k\sqrt{(1-\Lambda^2)^2 + (2\zeta\Lambda)^2}} cos(\omega_e t - \phi) \quad (5\text{-}43)$$

$$\phi = tan^{-1}\left\{\frac{2\zeta\Lambda}{1-\Lambda^2}\right\} \quad (5\text{-}44)$$

The amplitude of the response can be represented in dimensionless form by the so-called magnification factor (Q)

$$Q = \frac{Amplitude\ of\ oscilation}{Equivalent\ static\ displacement} = \frac{X_0}{F_0/K} = \frac{1}{\sqrt{(1-\Lambda^2)^2 + (2\zeta\Lambda)^2}} \quad (5\text{-}45)$$

Eq. (5-45) may be used to measure the amplitude and phase angle of the ship response in dimensionless format. This means that for a given forcing amplitude, F_0, the response amplitude changes depending on the damping factor and the tuning factor. The damping factor (ζ) relates to how much damping there is in the system. The larger the damping factor the smaller the magnification factor (Q). Increasing damping reduces the magnitude of the response. So, the tuning factor relates to how close the excitation frequency is to the natural frequency. When $\omega_e = \omega_n = 1$, in the absence of damping the response may go to infinity. The presence of damping reduces the response amplitude, but the max response will occur at $\Lambda = 1$. This peak is called **resonance**. Systems that are overdamped do not show any response amplitudes greater the static response. For over damped systems there is no magnification and no resonance.

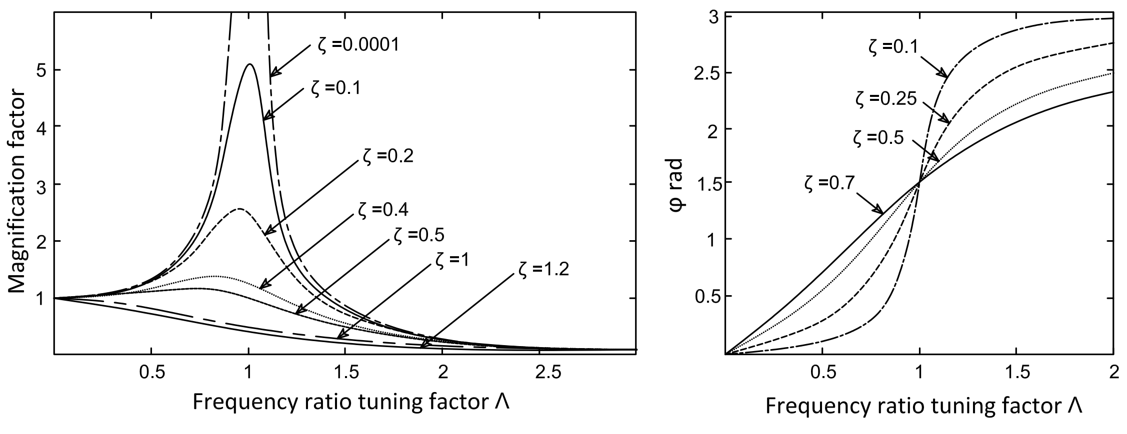

Figure 5-11 Magnification factor and phase angle representation

5. Response to random loading

Ship motions and sea loads arise from natural phenomena such as waves or turbulence and therefore cannot be adequately described by sinusoidal functions. The patterns of fluid actions do not repeat at regular intervals. Therefore, the right-hand side of the equation of motion can be expressed in a general form

$$M\ddot{x} + C\dot{x} + Kx = F(t) \tag{5-46}$$

In Eq (5-46) $F(t)$ is a stochastic excitation force attributed to waves. It is highly nonlinear and cannot adequately be expressed by sinusoidal or periodic functions. In such cases it is essential to resort to statistical analysis methods. According to Brouwers (2006) if we apply a *"Fourier integral"* on the excitation force, $F(t)$ we can define the external loading as

$$F(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_P(\omega)e^{i\omega t}d\omega \tag{5-47}$$

where $A_P(\omega)$ is the Fourier integral component which enables the time varying quantity, $\mathrm{F(t)}$, to be expressed in its frequency components ω. Hence, the response becomes,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} A_x(\omega)e^{i\omega t}d\omega \tag{5-48}$$

By substituting the response function and its derivatives into Equation (5-46) we obtain

$$\begin{aligned} &-m\int_{-\infty}^{\infty} A_x(\omega)\omega^2 e^{i\omega t}d\omega + c\int_{-\infty}^{\infty} A_x(\omega)i\omega e^{i\omega t}d\omega + k\int_{-\infty}^{\infty} A_x(\omega)e^{i\omega t}d\omega \\ &= \int_{-\infty}^{\infty} A_P(\omega)e^{i\omega t}d\omega \end{aligned} \tag{5-49}$$

$$\begin{aligned} &\int_{-\infty}^{\infty}[(-m\omega^2 + ci\omega + k)A_x(\omega) - A_p(\omega)]e^{i\omega t} = 0 \\ &\rightarrow [(-m\omega^2 + ci\omega + k)A_x(\omega) - A_p(\omega)]e^{i\omega t} = 0 \end{aligned} \tag{5-50}$$

Hence,

$$A_x(\omega) = \frac{A_P(\omega)}{-m\omega^2+ci\omega+k} \tag{5-51}$$

Equation (5-51) can be written in terms of the frequency ratio Λ

$$A_x(\Lambda) = \frac{A_P^n(\Lambda)}{-m\Lambda^2+\delta i\Lambda+1} \tag{5-52}$$

Where the parameter $\delta = \frac{c}{m\omega_n}$ determines the magnitude of damping; its effect will be analyzed in the resonance condition, Figure 5-14, and $A_P^n(\Lambda)$

$$A_P^n(\Lambda) = \frac{A_P(\omega)}{m\omega_n^2} \tag{5-53}$$

If we multiply this term with the complex conjugate, we get the spectral density for that system namely:

$$S_x(\omega) = \lim_{T\to\infty} \frac{1}{\pi T} |A_x(\omega)|^2 = \lim_{T\to\infty} \frac{1}{\pi T} \left| \frac{A_P^n(\Lambda)}{-m\Lambda^2 + \delta i \Lambda + 1} \right|^2 \rightarrow$$
$$S_x(\omega) = \frac{S_P^n}{(1-\Lambda^2)^2 + \delta^2 \Lambda^2} \tag{5-54}$$

As explained by Naess and Moan (2013), at **sub-critical case** (also known as quasi-static response) the system can reach high values of spectral density at small frequencies relative to the natural frequency and the stiffness has the major effect on the system. In such case the nominator of equation (5-55) nearly equals 1 and the spectral density becomes

$$S_x(\Lambda) = \frac{S_P^n(\Lambda)}{(1-\Lambda^2)^2 + \delta^2 \Lambda^2} \rightarrow S_x \approx S_P^n \tag{5-55}$$

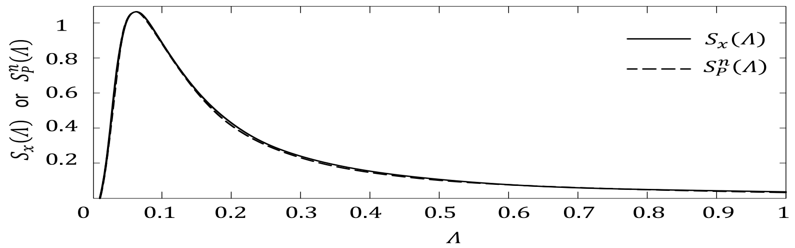

Figure 5-12 Excitation and response spectra of a typical Quasi-static response. $\Omega^* \ll 1$ ($\Omega^* = \Omega / \omega_0$ and Ω is the frequency where most excitation energy occurs)

At super-critical (also known as dynamic response) the highest values of spectral density lie in only high values of frequencies with respect to the natural frequency and damping plays an important role

$$S_x(\Lambda) = \frac{S_P^n(\Lambda)}{(1-\Lambda^2)^2 + \delta^2 \Lambda^2} \rightarrow S_x \approx \frac{S_P^n}{\Lambda^4} \tag{5-56}$$

In this case the term involving Λ^4 in the denominator will dominate, therefore, response is governed by inertia forces only.

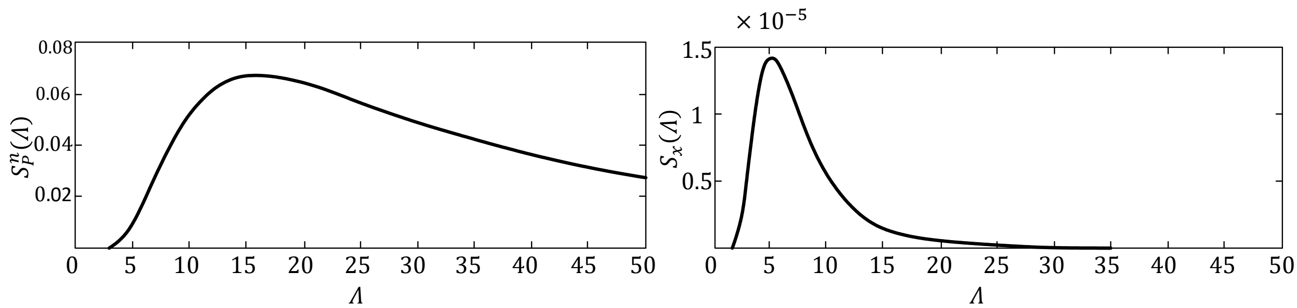

Figure 5-13 Excitation and response spectra of a typical dynamic response $\Omega^* \gg 1$

At resonance condition when there is very low damping the frequency ratio Λ approaches unity. The denominator approaches zero, and the spectral density approaches extremely large value:

$$S_x(\Lambda) = \frac{S_P^n(\Lambda)}{(1-\Lambda^2)^2+\delta^2\Lambda^2 \rightarrow \approx 0} \rightarrow S_x >>>> \qquad (5\text{-}57)$$

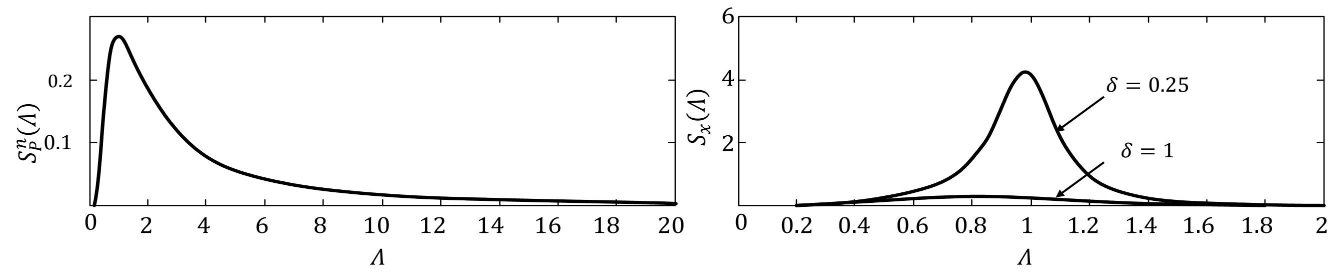

Figure 5-14 Excitation and response spectra of a typical resonance condition $\Omega^* = 1$

6. Questions

1. What is the equation of motion for a spring-mass-damper system subject to sinusoidal excitation? Derive motion amplitude, velocity, acceleration and phase.

2. Define the significance of underdamped, overdamped and critically damped dynamics for an 1-DOF spring-mass-damper system. Consequently, identify which of the seakeeping DOFs may be underdamped or overdamped.

3. How we can assess the damping coefficient of ships?

4. Describe the different ship coordinate systems and their use.

5. What is resonance and why it is relevant for seakeeping dynamics?

6. Explain the concept of hydrostatic stiffness, added mass and hydrodynamic damping within the context of ship dynamics.

7. What is meant by quasi static, dynamic and resonant responses?

8. Plot the displacement, velocity and acceleration of a free undamped system assuming $A = 1$, $\omega_n = 12$ rad/sec, $x_0 = 1\,m$ and $v_0 = 1$ m/s. What is the relationship between displacement, velocity and acceleration?

9. Plot the transient, steady state and the full solution against time for a damped system under harmonic excitation. Assume $A = X = 1;\ \omega = 2\,\omega_n = \pi;\ \ \varphi = \frac{\pi}{6}\ and\ \zeta = 0.1.$

10. Plot the amplitude ratio $X\,\omega_n^2\,/\,f_0$ and phase θ against frequency ratio for a damped system under harmonic excitation when $\zeta = 0.1, 0.25, 0.5\ and\ 0.7$. How does the damping factor affect the magnification factor?

7. References

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