

1. Proof of Exercise 2 Demo

Let A be a symmetric matrix with distinct eigenvalues. Show that the eigenvector matrix of A is orthogonal.

Let λ_i be the i th eigenvalue and v_i the corresponding eigenvector of a $p \times p$ matrix A . The goal is to show that $v_i^\top v_j = 0$, $i \neq j$. We can order the eigenvalues such that $\lambda_1 > \lambda_2 > \dots > \lambda_p$. The eigenvalues and -vectors satisfy:

$$\begin{cases} Av_i = \lambda_i v_i \\ Av_j = \lambda_j v_j. \end{cases}$$

First, we multiply the first equation with v_j^\top from the left side,

$$\begin{aligned} v_j^\top Av_i &= \lambda_i v_j^\top v_i \\ v_j^\top A^\top v_i &= \lambda_i v_j^\top v_i \\ (Av_j)^\top v_i &= \lambda_i v_j^\top v_i \\ \lambda_j v_j^\top v_i &= \lambda_i v_j^\top v_i \\ \Rightarrow (\lambda_j - \lambda_i) v_j^\top v_i &= 0. \end{aligned}$$

Since $\lambda_i \neq \lambda_j$, vectors v_j and v_i have to be orthogonal: $v_j^\top v_i = 0$, $i \neq j$. Hereby, the eigenvector matrix V of A satisfies $VV^\top = I$, if we choose the eigenvectors of A that have length 1.