1. Proof of Exercise 3 Demo

Show that the sample mean $T(\cdot)$ is affine equivariant.

Let X denote a $n \times p$ data matrix of n independent and identically distributed p-variate observations $x_1, x_2, \ldots, x_n$ from some continuous distribution with a finite covariance matrix Σ. Furthermore, consider the transformation,

$$y_i = Ax_i + b,$$

where A is a nonsingular $p \times p$ matrix and b is a p-variate location vector. Let $T(\cdot)$ be the sample mean. Then,

$$\begin{aligned} T(X) &= \frac{1}{n}\sum_{i=1}^{n} x_i, \\ T(Y) &= \frac{1}{n}\sum_{i=1}^{n} (Ax_i + b) = A\frac{1}{n}\sum_{i=1}^{n} x_i + \frac{1}{n}(nb) = AT(X) + b. \end{aligned}$$

Show that the sample covariance matrix $S(\cdot)$ is affine equivariant.

Let $S(\cdot)$ be the sample covariance matrix and consider the same transformation as in (a). Then,

$$\begin{aligned} S(X) &= \frac{1}{n-1}\sum_{i=1}^{n} (x_i - T(X))(x_i - T(X))^\top, \\ S(Y) &= \frac{1}{n-1}\sum_{i=1}^{n} (Ax_i + b - T(Y))(Ax_i + b - T(Y))^\top \\ &= \frac{1}{n-1}\sum_{i=1}^{n} (A(x_i - T(X)))(A(x_i - T(X)))^\top \\ &= A\frac{1}{n-1}\sum_{i=1}^{n} (x_i - T(X))(x_i - T(X))^\top A^\top = AS(X)A^\top. \end{aligned}$$