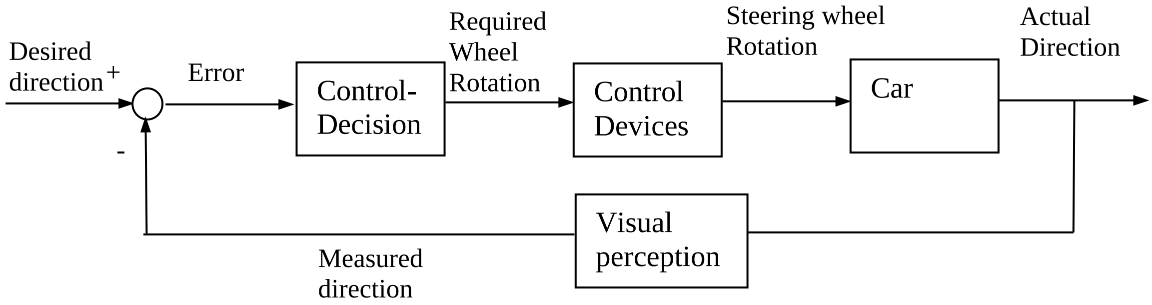
ELEC-C8201 Control and Automation
1. Exercise Answers

1.

a.



Control decision = Regulator
Control devices = Actuator
Car = process, systems, system
observation = sensors
Required Wheel rotation = Steering, reference value
Steering wheel rotation = action
Actual Direction = adjusted rate, response
Measured direction = measurements
Desired Direction = command, reference, Impulse

b. Examples of different types of disturbances:

Control decision: Fatigue, substance abuse, slowness of the driver and error estimates, other traffic
Control devices: Technical faults, constraints (wheel cannot turn in any arbitrary direction)
Car: Technical faults, tires, grooves on the wheel, sliding, weather
Visual perception: eyesight, glare, weather

2 a. The differential equation describing the system is derived force balance equation:

$$ma(t) + Bv(t) + kz(t) = F(t)$$
$$m\ddot{z}(t) + B\dot{z}(t) + kz(t) = F(t)$$
$$\ddot{z}(t) + \frac{B}{m}\dot{z}(t) + \frac{k}{m}z(t) = \frac{1}{m}F(t)$$

Substituting numerical values for the constants:

$$\ddot{z}(t) + 2\dot{z}(t) + z(t) = F(t) \qquad z(t) \propto y(t) \text{ ja } F(t) \propto u(t)$$

b. Select state variables such that they have a clear physical meaning (position and speed) as required by the task.

$$\begin{cases} x_1(t) = z(t) \\ x_2(t) = v(t) = \dot{z}(t) \end{cases}$$

From this definition, we obtain:

$$\dot{x}_1(t) = \dot{z}(t) = x_2(t) \quad \text{and} \quad y(t) = z(t) = x_1(t)$$

From the original differential equation, we get

$$\ddot{z}(t) = -z(t) - 2\dot{z}(t) + F(t)$$

which gives:

$$\ddot{z}(t) = \dot{x}_2(t) = -z(t) - 2\dot{z}(t) + F(t) = -x_1(t) - 2x_2(t) + u(t)$$

Putting these together:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t) - 2x_2 + u(t) \\ y(t) = x_1(t) \end{cases}$$

Writing as state space representation:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

3.

a. The basic equation of the rotation

$$\sum_i T_i = J \frac{d^2\theta(t)}{dt^2}$$

The sum of the torque that will affect the axis of rotation shall be equal to the product of inertia and angular acceleration. The task in the case of the antenna is to turn with the effect of the torque $T(t)$ and motion resisting force $B\dot{\theta}$. The differential equation is therefore:

$$T(t) - B\dot{\theta}(t) = J\ddot{\theta}(t) \Leftrightarrow \ddot{\theta}(t) = -\frac{B}{J}\dot{\theta}(t) + \frac{1}{J}T(t)$$

b. The antenna angle and angular speed are selected as state variables as per the given instructions:

$$\begin{cases} x_1(t) = \theta(t) \\ x_2(t) = \dot{\theta}(t) \end{cases}$$

The derivatives of the state variables are obtained from the above differential equations as:

$$\begin{cases} \dot{x}_1(t) = \dot{\theta}(t) = x_2 \\ \dot{x}_2(t) = \ddot{\theta}(t) = -\frac{B}{J}\dot{\theta}(t) + \frac{1}{J}T(t) = -\frac{B}{J}x_2(t) + \frac{1}{J}T(t) \end{cases}$$

Rewriting in the state space form with output $y(t)$ as the angular position of the antenna:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{B}{J} \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} T(t) \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(t) \end{cases}$$

c. The system is controlled so that the torque produced is proportional to the difference between the actual antenna angle θ(t) and the desired angle θ_{ref}(t) according the relation:

$$T(t) = k\left(\theta_{ref}(t) - \theta(t)\right)$$

where k is scalar and standard. Placing the equation above in a differential equation derived from **(a)** is achieved by

$$k\left(\theta_{ref}(t) - \theta(t)\right) = J\ddot{\theta}(t) + B\dot{\theta}(t)$$
$$\Rightarrow \ddot{\theta}(t) = -\frac{B}{J}\dot{\theta}(t) - \frac{k}{J}\theta(t) + \frac{k}{J}\theta_{ref}(t)$$

Proceeding like in section (b), we get,

$$\begin{cases} x_1(t) = \theta(t) \\ x_2(t) = \dot{\theta}(t) \end{cases} \Rightarrow \begin{cases} \dot{x}_1(t) = \dot{\theta}(t) = x_2 \\ \dot{x}_2(t) = \ddot{\theta}(t) = -\frac{B}{J}x_2(t) - \frac{k}{J}x_1(t) + \frac{k}{J}\theta_{ref}(t) \end{cases}$$

Based on the differential equations, the matrix representation is given by:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\frac{k}{J} & -\frac{B}{J} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ \frac{k}{J} \end{bmatrix} \theta_{ref}(t) \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

Note that this SS representation represents an already closed loop system. The input is now the desired angular deviation value (reference value). Previous models (differential equations and state presentations) only described the process being investigated without any feedback control.

4. Note. (Laplace transformation): In part **1**II. solve the tasks a, b and c and in **1**. III. solve tasks a, b and c. Note. In the solutions, these are presented in quite compact form. So, make sure you know how to compute them yourself.

Solutions:

These are some of the explanations to the given tasks:

II.a. Examine the denominator. It is real and order-3. The expression is obtained directly from a form available in the inverse Laplace table.

II.b. The denominator has real roots here as well. A partial fraction is derived as follows:

$$F(s) = \frac{4}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3} = \frac{A(s+3)+B(s+1)}{(s+1)(s+3)}$$
$$\Rightarrow As + 3A + Bs + B = 4$$
$$\Rightarrow (A+B)s + 3A + B - 4 = 0$$
$$\Rightarrow A + B = 0,\ 3A + B - 4 = 0$$
$$\Rightarrow A = 2, \quad B = -2$$

$$f(t) = \left(2e^{-t} - 2e^{-3t}\right) u_S(t)$$

II.c. This is given by. $F(s) = \dfrac{10s+8}{s(s+1)(s+2)}$. Again, use partial fraction method, but determining the coefficients for the partial fractions using the Heaviside method:

The formula is initially difficult to grasp, but it is easy to memorize. For example, if

$$\frac{1}{(s-a)^n(s-b)} = \frac{A_0}{(s-a)^n} + \frac{A_1}{(s-a)^{n-1}} + \mathrm{K} + \frac{A_{n-1}}{(s-a)} + \frac{B}{s-b}, \quad n \in N$$

Commonly $A_i = \lim_{s \to a} \left\{ \frac{1}{i!} \frac{d^{(i)}}{di^n} \left[(s-a)^n \frac{1}{(s-a)^n (s-b)} \right] \right\}$, B the same way.

So, if there are multiple poles, a separate partial function must be taken for each of them. Note the (!) and the order of the derivative. In the given task:

$$F(s) = \frac{10s+8}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$A = \lim_{s \to 0} \frac{10s+8}{(s+1)(s+2)} = 4$$

$$B = \lim_{s \to -1} \frac{10s+8}{s(s+2)} = 2$$

$$C = \lim_{s \to -2} \frac{10s+8}{s(s+1)} = -6$$

$$f(t) = \left(4 + 2e^{-t} - 6e^{-2t}\right) u_S(t)$$

If the denominator has an imaginary roots, then use the method for D-F in the same way.

III. a, b, c. Can be found directly in tables.

All solution answers:

4.

i. According to the definition of Laplace transformation:

$$F(s) = \int_0^\infty f(t) e^{-st} dt$$

a. $F(s) = \int_0^\infty A e^{-st} dt = -\frac{A}{s} \int_0^\infty -s e^{-st} dt = -\frac{A}{s} \Big|_0^\infty e^{-st} = -\frac{A}{s}[0-1] = \frac{A}{s}$

b. $F(s) = \int_0^\infty e^{-at} e^{-st} dt = \int_0^\infty e^{-(a+s)t} dt = \mathrm{K} = \frac{1}{s+a}$

ii. Sections II and III need Laplace transformation and Laplace inverse transformation. It can be done with basic transformations in the transformation tables. A function in the time domain that is desired in the Laplace domain is arranged in a form matching the form found in tables.

a. $F(s) = \frac{4}{(s+3)^2}$

$f(t) = 4te^{-3t} u_S(t)$

b. $F(s)=\dfrac{4}{(s+1)(s+3)}=\dfrac{A}{s+1}+\dfrac{B}{s+3}=\dfrac{2}{s+1}-\dfrac{2}{s+3}$

$f(t)=\left(2e^{-t}-2e^{-3t}\right)u_s(t)$

c. $F(s)=\dfrac{A}{s}+\dfrac{B}{s+1}+\dfrac{C}{s+2}=\dfrac{4}{s}+\dfrac{2}{s+1}+\dfrac{-6}{s+2}$

$f(t)=\left(4+2e^{-t}-6e^{-2t}\right)u_s(t)$

d. $F(s)=\dfrac{10(s+2)}{(s+2)^2+9}+\dfrac{30}{(s+2)^2+9}$

$f(t)=10e^{-2t}(\cos 3t+\sin 3t)u_s(t)$

e. $F(s)=\dfrac{\frac{4}{2\sqrt{2}}\cdot 2\sqrt{2}}{s^2+\left(2\sqrt{2}\right)^2}$

$f(t)=\left(\sqrt{2}\sin 2\sqrt{2}t\right)u_s(t)$

f. $F(s)=\dfrac{As+B}{s^2+16}+\dfrac{Cs+D}{s^2+4s+20}$

$16s+16\equiv A\left(s^3+4s^2+20s\right)+B\left(s^2+4s+20\right)+C\left(s^3+16s\right)+D\left(s^2+16\right)$

$\Rightarrow A=0, C=0, B=4, D=-4$

$F(s)=\dfrac{4}{s^2+16}-\dfrac{4}{(s+2)^2+16}$

$f(t)=\left[\left(1-e^{-2t}\right)\sin 4t\right]u_s(t)$

iii.

a. $f(t)=2\cdot 1+3\cdot t+8\cdot\dfrac{t^2}{2}-2\cdot e^{-3t}$

$F(s)=\dfrac{2}{s}+\dfrac{3}{s^2}+\dfrac{8}{s^3}-\dfrac{2}{s+3}=\dfrac{9s^2+17s+24}{s^3(s+3)}$

b. $f(t)=3\cdot\dfrac{t^1e^{-4t}}{1}+2\cdot 1-2\cdot e^{-4t}$

$F(s)=\dfrac{3}{(s+4)^2}+\dfrac{2}{s}-\dfrac{2}{s+4}=\dfrac{11s+32}{s(s+4)^2}$

c. $f(t) = 4 \cdot \sin 2t + 5 \cdot \cos 2t$

$$F(s) = \frac{4 \cdot 2}{s^2 + 4} + \frac{5s}{s^2 + 4} = \frac{5s + 8}{s^2 + 4}$$

d. If $L\{f(t)u_s(t)\} = F(s)$, then

$$L\{f(t-a)u_s(t-a)\} = e^{-as} F(s)$$

$$F(s) = \frac{2}{s^2} e^{-s}$$

e. $f(t) = 4 \cdot (e^{-3t} \sin 2t) + 4 \cdot (e^{-3t} \cos 2t)$

$$F(s) = \frac{4 \cdot 2}{(s+3)^2 + 4} + \frac{4(s+3)}{(s+3)^2 + 4} = \frac{4s + 20}{s^2 + 6s + 13}$$