



# Discrete-time optimal control

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## Overview

We combine the notions on dynamic systems and simulation with the notions on nonlinear programming, to formulate a general **discrete-time optimal control** problem

- We understand and treat them as special forms of nonlinear programs

## Overview (cont.)

Consider a system  $f$  which maps an initial state vector  $x_k$  onto a final state vector  $x_{k+1}$

- We also consider the presence of a control  $u_k$  that modifies the transition

$$x_{k+1} = f(x_k, u_k | \theta_x), \quad (k = 0, 1, \dots, K-1)$$

We consider transitions over a time-horizon, from time  $k = 0$  to time  $k = K$

$$0 \cdots 1 \cdots \cdots (k-1) \cdots k \cdots (k+1) \cdots \cdots (K-1) \cdots K$$

Over the time-horizon of interest, we thus have the sequences

- ~ States  $\{x_k\}_{k=0}^K$ , with  $x_k \in \mathcal{R}^{N_x}$
- ~ Controls  $\{u_k\}_{k=0}^{K-1}$ , with  $u_k \in \mathcal{R}^{N_u}$

For notational simplicity, we used time-invariant dynamics  $f$

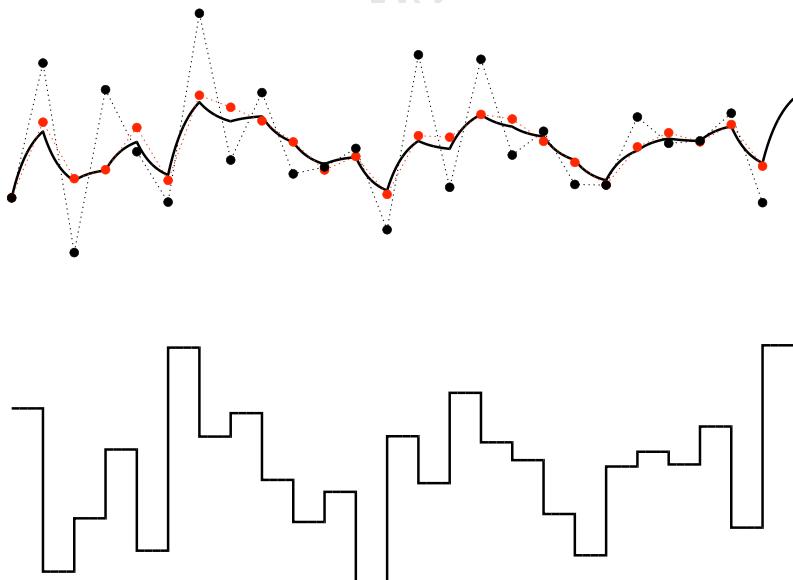
- In general, we have  $x_{k+1} = f_k(x_k, u_k | \theta_x)$

## Overview (cont.)

$$x_{k+1} = f(x_k, u_k | \theta_x), \quad (k = 0, 1, \dots, K-1)$$

The dynamics  $f$  are often derived from the discretisation of a continuous-time system

- As result of a numerical integration schemes, under piecewise constant controls



## Overview (cont.)

$$x_{k+1} = f(x_k, u_k | \theta_x), \quad (k = 0, 1, \dots, K-1)$$

Given an initial state  $x_0$  and any sequence of controls  $\{u_k\}_{k=0}^{K-1}$ , we know all the states

The forward simulation function determines the sequence of states  $\{x_k\}_{k=0}^K$

$$\begin{aligned} f_{\text{sim}} : \mathcal{R}^{N_x + (K \times N_u)} &\rightarrow \mathcal{R}^{(K+1)N_x} \\ &: (x_0, u_0, u_1, \dots, u_{K-1}) \mapsto (x_0, x_1, \dots, x_K) \end{aligned}$$


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For arbitrary systems, the forward simulation map is built recursively

$$\begin{aligned} x_0 &= x_0 \\ x_1 &= f(x_0, u_0) \\ x_2 &= f(x_1, u_1) \\ &= f(f(x_0, u_0), u_1) \\ x_3 &= f(x_2, u_2) \\ &= f(f(f(x_0, u_0), u_1), u_2) \\ \dots &= \dots \end{aligned}$$

## Overview (cont.)

$$x_{k+1} = f(x_k, u_k | \theta_x), \quad (k = 0, 1, \dots, K-1)$$

In optimal control, the dynamics can be used as equality constraints in optimisation

In this case, the initial state vector  $x_0$  is not necessarily known, or fixed

- It can be one of the decision variables to be determined
- Moreover, certain constraints would apply to it

Similarly, also the final state  $x_K$  can be treated as decision variable in an optimisation

## Formulations

Simultaneous approach

Sequential approach

## Overview (cont.)

### Initial and terminal state constraints

We express the constraints on initial and terminal states in terms of function  $r(x_0, x_K)$

$$r : \mathcal{R}^{N_x + N_x} \rightarrow \mathcal{R}^{N_r}$$

We express the desire to reach certain initial and terminal states as equality constraints

$$r(x_0, x_K) = 0$$

For fixed initial state  $x_0 = \bar{x}_0$ , we have

$$r(x_0, x_K) = x_0 - \bar{x}_0$$

For fixed terminal state  $x_K = \bar{x}_K$ , we have

$$r(x_0, x_K) = x_K - \bar{x}_K$$

For fixed both initial and terminal states,  $x_0 = \bar{x}_0$  and  $x_K = \bar{x}_K$ , we have

$$r(x_0, x_K) = \begin{bmatrix} x_0 - \bar{x}_0 \\ x_K - \bar{x}_K \end{bmatrix}$$

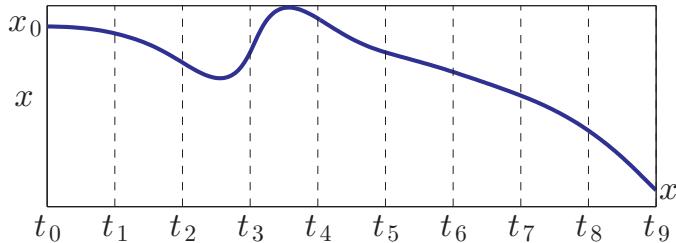
## Formulations

Simultaneous approach

Sequential approach

## Overview (cont.)

For fixed both initial and terminal states,  $x_0 = \bar{x}_0$  and  $x_K = \bar{x}_K$ , we have



$$r(x_0, x_K) = \underbrace{\begin{bmatrix} x_0^{(1)} - \bar{x}_0^{(1)} \\ x_0^{(2)} - \bar{x}_0^{(2)} \\ \vdots \\ x_0^{(N_x)} - \bar{x}_0^{(N_x)} \end{bmatrix}}_{N_r \times 1} \quad \begin{bmatrix} x_K^{(1)} - \bar{x}_K^{(1)} \\ x_K^{(2)} - \bar{x}_K^{(2)} \\ \vdots \\ x_K^{(N_x)} - \bar{x}_K^{(N_x)} \end{bmatrix}_{N_r \times 1}$$

## Overview (cont.)

### Path constraints

We can express certain constraints on arbitrary state and control values,  $x_k$  and  $u_k$

- These constraints often represent certain technological restrictions
- They are expressed in terms of inequality constraints
- The main idea is to use them to avoid violations

$$h(x_k, u_k) \leq 0, \quad k = 0, 1, \dots, K - 1$$

For notational simplicity, we used time-invariant inequality constraint functions  $h$

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For upper and lower bounds on the controls,  $u_{\min} \leq u_k \leq u_{\max}$ , we have

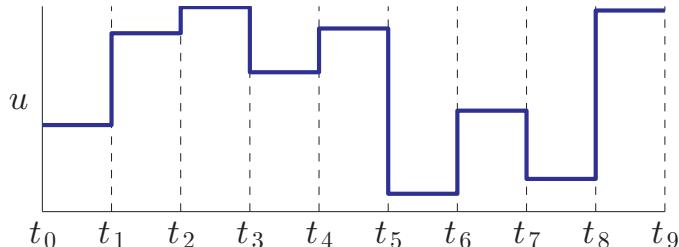
$$h(x_k, u_k) = \begin{bmatrix} u_k - u_{\max} \\ u_{\min} - u_k \end{bmatrix}$$

For upper and lower bounds on the states,  $x_{\min} \leq x_k \leq x_{\max}$ , we have

$$h(x_k, u_k) = \begin{bmatrix} x_k - x_{\max} \\ x_{\min} - x_k \end{bmatrix}$$

## Overview (cont.)

For upper and lower bounds on the controls,  $u_{\min} \geq u_k \geq u_{\max}$ , we have



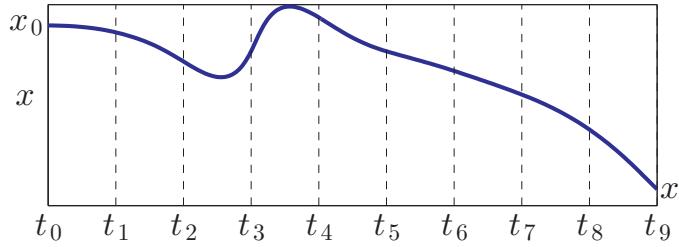
$$h(x_k, u_k) = \begin{bmatrix} u_k^{(1)} - u_{\max}^{(1)} \\ u_k^{(2)} - u_{\max}^{(2)} \\ \vdots \\ u_k^{(N_u)} - u_{\max}^{(N_u)} \\ \hline u_{\min}^{(1)} - u_k^{(1)} \\ u_{\min}^{(2)} - u_k^{(2)} \\ \vdots \\ u_{\min}^{(N_u)} - u_k^{(N_u)} \end{bmatrix}$$

## Overview (cont.)

### Formulations

- Simultaneous approach
- Sequential approach

For upper and lower bounds on the states,  $x_{\min} \geq x_k \geq x_{\max}$ , we have



$$h(x_k, u_k) = \begin{bmatrix} x_k^{(1)} - x_{\max}^{(1)} \\ x_k^{(2)} - x_{\max}^{(2)} \\ \vdots \\ x_k^{(N_x)} - x_{\max}^{(N_x)} \\ \\ x_{\min}^{(1)} - x_k^{(1)} \\ x_{\min}^{(2)} - x_k^{(2)} \\ \vdots \\ x_{\min}^{(N_x)} - x_k^{(N_x)} \end{bmatrix}$$

### Formulations

- Simultaneous approach
- Sequential approach

## Problem formulations

Discrete-time optimal control

## Problem formulations

### Formulations

Simultaneous approach  
Sequential approach

We have the system dynamics and the specifications on the state and control constraints

We use them to formulate the control problem, as constrained nonlinear optimisation

$$\begin{aligned} & \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ & \text{subject to } x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\ & \quad h(x_k, u_k) \leq 0, \quad k = 0, 1, \dots, K-1 \\ & \quad r(x_0, x_K) = 0 \end{aligned}$$

### Formulations

Simultaneous approach  
Sequential approach

$$\begin{aligned} & \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ & \text{subject to } x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\ & \quad h(x_k, u_k) \leq 0, \quad k = 0, 1, \dots, K-1 \\ & \quad r(x_0, x_K) = 0 \end{aligned}$$

The **objective function**, two terms

$$\sum_{k=0}^{K-1} L(x_k, u_k) + E(x_K)$$

The **decision variables**, two sets

$$\begin{aligned} & x_0, x_1, \dots, x_{K-1}, x_K \\ & u_0, u_1, \dots, u_{K-1} \end{aligned}$$

The **equality constraints**, two sets

$$\begin{aligned} & x_{k+1} - f(x_k, u_k | \theta_x) = 0 \quad (k = 0, \dots, K-1) \\ & r(x_0, x_K) = 0 \end{aligned}$$

The **inequality constraints**

$$h(x_k, u_k) \leq 0 \quad (k = 0, 1, \dots, K-1)$$

**Formulations**  
 Simultaneous approach  
 Sequential approach

## Problem formulations (cont.)

$$\begin{aligned} & \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ \text{subject to } & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\ & h(x_k, u_k) \leq 0, \quad k = 0, 1, \dots, K-1 \\ & r(x_0, x_K) = 0 \end{aligned}$$

The **objective function** is the sum of all **stage costs**  $L(x_k, u_k)$  and a **terminal cost**  $E(x_K)$

$$\underbrace{\sum_{k=0}^{K-1} L(x_k, u_k) + E(x_K)}_{f(w) \in \mathcal{R}}$$

That is,

$$L(x_0, u_0) + L(x_1, u_1) + \dots + L(x_{K-1}, u_{K-1}) + E(x_K)$$

Stage cost is a (potentially nonlinear and time-varying) function of state and controls

The **decision variables**,  $K \times N_u$  **control** and  $(K+1) \times N_x$  **state variables**

$$\underbrace{(x_0, x_1, \dots, x_{K-1}, x_K) \cup (u_0, u_1, \dots, u_{K-1})}_{w \in \mathcal{R}^{K \times N_u + (K+1) \times N_x}}$$

**Formulations**  
 Simultaneous approach  
 Sequential approach

## Problem formulations (cont.)

$$\begin{aligned} & \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ \text{subject to } & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\ & h(x_k, u_k) \leq 0, \quad k = 0, 1, \dots, K-1 \\ & r(x_0, x_K) = 0 \end{aligned}$$

The **equality constraints**, the  $K$  **dynamics** and the  $N_r$  **boundary conditions**

$$\underbrace{x_{k+1} - f(x_k, u_k | \theta_x) = 0 \quad (k = 0, \dots, K-1)}_{g(w) \in \mathcal{R}^{N_g}} \\ r(x_0, x_K) = 0$$

The **inequality constraints**

$$\underbrace{h(x_k, u_k) \leq 0 \quad (k = 0, 1, \dots, K-1)}_{h(w) \in \mathcal{R}^{N_h}}$$

## Problem formulations (cont.)

### Formulations

Simultaneous  
approach

Sequential approach

$$\begin{aligned} \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ \text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\ & h(x_k, u_k) \leq 0, \quad k = 0, 1, \dots, K-1 \\ & r(x_0, x_K) = 0 \end{aligned}$$

The discrete-time optimal control problem is a potentially very large nonlinear program

- In principle, its solution can be approached using any generic NLP solver

We discuss the two approaches used to solve discrete-time optimal control problems

- The **simultaneous approach**
- The **sequential approach**

### Formulations

Simultaneous  
approach

Sequential approach

## The simultaneous approach

### Problem formulations

## Problem formulations | Simultaneous approach

Formulations

Simultaneous approach

Sequential approach

$$\begin{aligned} \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ \text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\ & h(x_k, u_k) \leq 0, \quad k = 0, 1, \dots, K-1 \\ & r(x_0, x_K) = 0 \end{aligned}$$

The **simultaneous approach** solves the problem in the space of all the decision vars

$$w = (x_0, u_0, x_1, u_1, \dots, x_{K-1}, u_{K-1}, x_K)$$

Thus, there are  $(K \times N_u) + ((K + 1) \times N_x)$  decision variables

## Problem formulations | Simultaneous approach

Formulations

Simultaneous approach

Sequential approach

The Lagrangian function of the problem,

$$\mathcal{L}(w, \lambda, \mu) = f(w) + \lambda^T g(w) + \mu^T h(w)$$

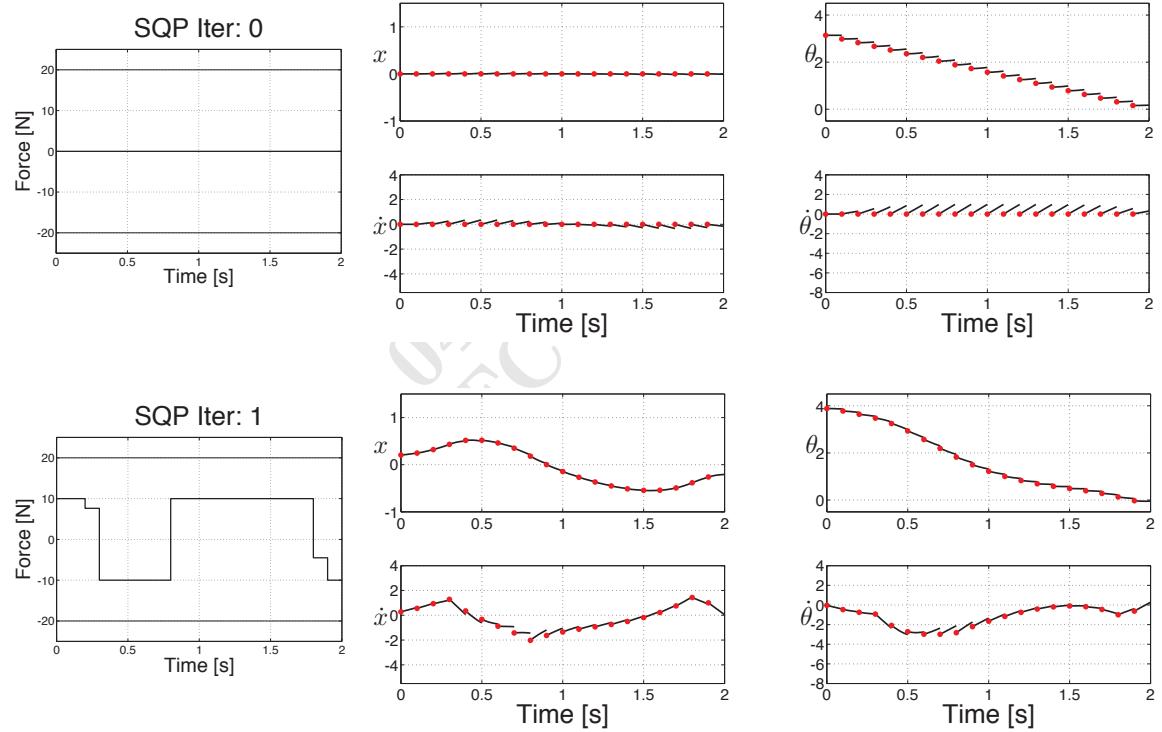
The Karush-Kuhn-Tucker conditions,

$$\begin{aligned} \nabla f(w^*) + \nabla g(w^*) \lambda^* + \nabla h(w^*) \mu^* &= 0 \\ g(w^*) &= 0 \\ h(w^*) &\leq 0 \\ \mu^* &\geq 0 \\ \mu_{n_h}^* h_{n_h}(w^*) &= 0, \quad n_h = 1, \dots, N_h \end{aligned}$$

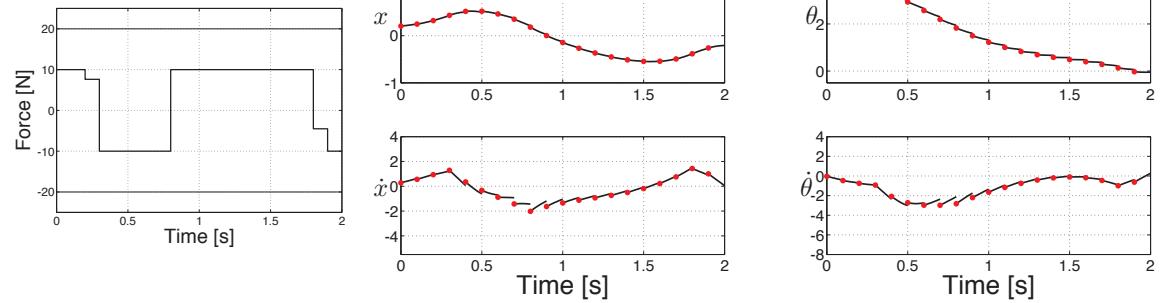
If point  $w^* = (x_0^*, u_0^*, \dots, x_{K-1}^*, u_{K-1}^*, x_K^*)$  is a local minimiser of the nonlinear program and if LICQ holds at  $w^*$ , there exist two vectors, the Lagrange multipliers  $\lambda \in \mathbb{R}^{N_g}$  and  $\mu \in \mathbb{R}^{N_h}$ , such that the Karush-Kuhn-Tucker conditions are verified

## Problem formulations | Simultaneous approach (cont.)

Formulations  
Simultaneous approach  
Sequential approach

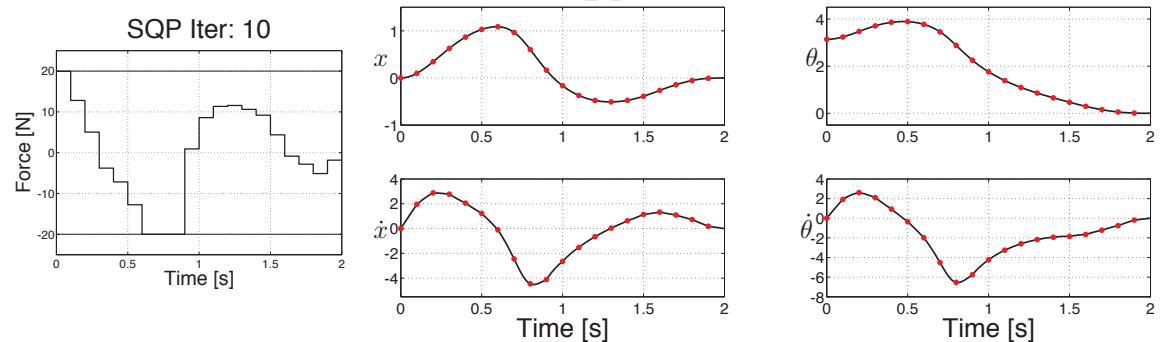


SQP Iter: 1



## Problem formulations | Simultaneous approach (cont.)

Formulations  
Simultaneous approach  
Sequential approach



## Problem formulations | Simultaneous approach (cont.)

## Formulations

Simultaneous  
approach

Sequential approach

To understand more closely the structure and sparsity properties, consider an example

$$\begin{aligned} \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ \text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\ & r(x_0, x_K) = 0 \end{aligned}$$

This optimal control problem in discrete-time has no inequality constraints

- Inequality constraints are omitted for notational simplicity

The objective  $f(w) = E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k)$  of the decision variables,

$$w = (x_0, u_0, x_1, u_1, \dots, x_{K-1}, u_{K-1}, x_K)$$

## Problem formulations | Simultaneous approach (cont.)

## Formulations

Simultaneous  
approach

Sequential approach

$$\begin{aligned} \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ \text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\ & r(x_0, x_K) = 0 \end{aligned}$$

We define the equality constraint function by concatenation

$$\begin{aligned} g(w) &= \begin{bmatrix} g_1(w) \\ g_2(w) \\ \vdots \\ g_{N_g}(w) \end{bmatrix} \\ &= \underbrace{\begin{bmatrix} x_1 - f(x_0, u_0) \\ x_2 - f(x_1, u_1) \\ \vdots \\ x_K - f(x_{K-1}, u_{K-1}) \end{bmatrix}}_{((K \times N_x) + N_r) \times 1} \\ &\quad \underbrace{r(x_0, x_K)}_{(K \times N_x) + N_r \times 1} \end{aligned}$$

## Problem formulations | Simultaneous approach (cont.)

## Formulations

Simultaneous

approach

Sequential approach

$$\begin{aligned} & \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ \text{subject to } & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\ & r(x_0, x_K) = 0 \end{aligned}$$

The Lagrangian function for equality constrained problems,

$$\mathcal{L}(w) = f(w) + \lambda^T g(w)$$

The equality multipliers,

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_K, \lambda_{N_r})$$

The KKT conditions,

$$\begin{aligned} \nabla_w \mathcal{L}(w, \lambda) &= 0 \\ g(w) &= 0 \end{aligned}$$

## Problem formulations | Simultaneous approach (cont.)

## Formulations

Simultaneous

approach

Sequential approach

$$\underbrace{\begin{bmatrix} \lambda_1 & \lambda_2 & \dots & \lambda_K & \lambda_{N_r} \end{bmatrix}}_{\lambda^T} \underbrace{\begin{bmatrix} x_1 - f(x_0, u_0) \\ x_2 - f(x_1, u_1) \\ \vdots \\ x_K - f(x_{K-1}, u_{K-1}) \\ r(x_0, x_K) \end{bmatrix}}_{g(w)}$$

After expanding the terms in the inner product, we re-write the Lagrangian function

$$\begin{aligned} \mathcal{L}(w, \lambda) = & \\ & \underbrace{E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k)}_{f(w)} + \underbrace{\left( \sum_{k=0}^{K-1} \lambda_{k+1}^T (f(x_k, u_k) - x_{k+1}) + \lambda_{N_r}^T r(x_0, x_K) \right)}_{\lambda^T g(w)} \end{aligned}$$

## Problem formulations | Simultaneous approach (cont.)

## Formulations

Simultaneous approach

Sequential approach

Consider one of the dynamic constraints,

$$x_{k+1} - f(x_k, u_k) = 0$$

More explicitly, we have

$$\underbrace{\begin{bmatrix} x_{k+1}^{(1)} - f_1(x_k, u_k) \\ x_{k+1}^{(2)} - f_2(x_k, u_k) \\ \vdots \\ x_{k+1}^{(n_x)} - f_{n_x}(x_k, u_k) \\ \vdots \\ x_{k+1}^{(N_x)} - f_{N_x}(x_k, u_k) \end{bmatrix}}_{N_x \times 1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

## Problem formulations | Simultaneous approach (cont.)

## Formulations

Simultaneous approach

Sequential approach

Consider the corresponding product with the equality multiplier,

$$\underbrace{\lambda_{k+1}^T \underbrace{(f(x_k, u_k) - x_{k+1})}_{N_x \times 1}}_{1 \times 1}$$

More explicitly, we have

$$\underbrace{\begin{bmatrix} \lambda_{k+1}^{(1)} & \lambda_{k+1}^{(2)} & \cdots & \lambda_{k+1}^{(n_x)} & \cdots & \lambda_{k+1}^{(N_x)} \end{bmatrix}}_{1 \times N_x} \underbrace{\begin{bmatrix} x_{k+1}^{(1)} - f_1(x_k, u_k) \\ x_{k+1}^{(2)} - f_2(x_k, u_k) \\ \vdots \\ x_{k+1}^{(n_x)} - f_{n_x}(x_k, u_k) \\ \vdots \\ x_{k+1}^{(N_x)} - f_{N_x}(x_k, u_k) \end{bmatrix}}_{N_x \times 1}$$

## Problem formulations | Simultaneous approach (cont.)

## Formulations

### Simultaneous approach

## Sequential approach

Similarly, consider the boundary constraint,

$$r(x_0, x_K) = 0$$

In more detail, we have,

$$r(x_0, x_N) = \begin{bmatrix} x_0^{(1)} - \bar{x}_0^{(1)} \\ x_0^{(2)} - \bar{x}_0^{(2)} \\ \vdots \\ x_0^{(N_x)} - \bar{x}_0^{(N_x)} \\ \hline x_K^{(1)} - \bar{x}_K^{(1)} \\ x_K^{(2)} - \bar{x}_K^{(2)} \\ \vdots \\ x_K^{(N_x)} - \bar{x}_K^{(N_x)} \end{bmatrix}_{N_r \times 1}$$

Problem formulations | Simultaneous approach (cont.)

For the product  $\lambda_{N_r}^T r(x_0, x_K)$  with the equality multiplier, we have

## Formulations

### Simultaneous approach

## Sequential approach

$$\underbrace{\lambda_{N_r}^T}_{1 \times N_r} \underbrace{r(x_0, x_K)}_{N_r \times 1} \underbrace{}_{1 \times 1}$$

More explicitly, we have

$$\underbrace{\begin{bmatrix} \lambda_{N_r}^{(1)} & \dots & \lambda_{N_r}^{(N_x)} & \lambda_{N_r}^{(N_x+1)} & \dots & \lambda_{N_r}^{(2N_x)} \end{bmatrix}}_{1 \times N_r} \quad \underbrace{\begin{bmatrix} x_0^{(1)} - \bar{x}_0^{(1)} \\ x_0^{(2)} - \bar{x}_0^{(2)} \\ \vdots \\ x_0^{(N_x)} - \bar{x}_0^{(N_x)} \end{bmatrix}}_{N_r \times 1}$$

## Problem formulations | Simultaneous approach (cont.)

For the Lagrangian function for equality constrained problems, we thus have

$$\mathcal{L}(w, \lambda) = \underbrace{f(w)}_{1 \times 1} + \underbrace{\begin{bmatrix} \underbrace{\lambda_1}_{1 \times N_x} & \underbrace{\lambda_2}_{1 \times N_x} & \cdots & \underbrace{\lambda_K}_{1 \times N_x} & \underbrace{\lambda_{N_r}}_{1 \times N_r} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda^T & & & & \end{bmatrix}}_{1 \times ((K \times N_x) + N_r)} \underbrace{\begin{bmatrix} x_1 - f(x_0, u_0) \\ x_2 - f(x_1, u_1) \\ \vdots \\ x_K - f(x_{K-1}, u_{K-1}) \\ r(x_0, x_K) \end{bmatrix}}_{((K \times N_x) + N_r) \times 1}}_{g(w)}$$

## Problem formulations | Simultaneous approach (cont.)

$$\nabla_w \mathcal{L}(w, \lambda) = 0$$

$$g(w) = 0$$

The second KKT condition,

$$x_{k+1} - f(x_k, u_k) = 0 \quad (k = 0, \dots, K-1)$$

$$r(x_0, x_K) = 0$$

The first KKT condition regards the derivative of  $\mathcal{L}$  with respect to the primal vars  $w$

$$w = (x_0, u_0, x_1, u_1, \dots, x_{K-1}, u_{K-1}, x_K)$$

The Lagrangian function in structural form,

$$\underbrace{E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k)}_{f(w)} + \underbrace{\left( \sum_{k=0}^{K-1} \lambda_{k+1}^T (f(x_k, u_k) - x_{k+1}) + \lambda_{N_r}^T r(x_0, x_K) \right)}_{\lambda^T g(w)} \underbrace{\mathcal{L}(w, \lambda)}_{\mathcal{L}(w, \lambda)}$$

## Problem formulations | Simultaneous approach (cont.)

Formulations

Simultaneous  
approach

Sequential approach

$$g(w) = 0$$

For the second KKT condition, we have

$$\begin{aligned} x_{k+1} - f(x_k, u_k) &= 0 \quad (k = 0, \dots, K-1) \\ r(x_0, x_K) &= 0 \end{aligned}$$

That is,

$$\left[ \begin{array}{c} \overbrace{x_1 - f(x_0, u_0)}^{N_x \times 1} \\ \overbrace{x_2 - f(x_1, u_1)}^{N_x \times 1} \\ \vdots \\ \overbrace{x_K - f(x_{K-1}, u_{K-1})}^{N_x \times 1} \\ \hline r(x_0, x_K) \end{array} \right] = \left[ \begin{array}{c} \underbrace{0}_{N_x \times 1} \\ \underbrace{0}_{N_x \times 1} \\ \vdots \\ \underbrace{0}_{N_x \times 1} \\ \hline \underbrace{0}_{N_r \times 1} \end{array} \right]$$

## Problem formulations | Simultaneous approach (cont.)

Formulations

Simultaneous  
approach

Sequential approach

$$\nabla_w \mathcal{L}(w, \lambda) = 0$$

Consider the gradient of the Lagrangian function, it is a concatenation of gradients

$$\nabla_w \mathcal{L}(w, \lambda) = \left[ \begin{array}{c} \nabla_{x_0} \mathcal{L}(w, \lambda) \\ \nabla_{x_1} \mathcal{L}(w, \lambda) \\ \vdots \\ \nabla_{x_K} \mathcal{L}(w, \lambda) \\ \hline \nabla_{u_0} \mathcal{L}(w, \lambda) \\ \nabla_{u_1} \mathcal{L}(w, \lambda) \\ \vdots \\ \nabla_{u_{K-1}} \mathcal{L}(w, \lambda) \end{array} \right]$$

For the second KKT conditions, it is necessary to determine/evaluate the derivatives

## Problem formulations | Simultaneous approach (cont.)

$$E(x_K) + \underbrace{\sum_{k=0}^{K-1} L(x_k, u_k) + \left( \sum_{k=0}^{K-1} \lambda_{k+1}^T (f(x_k, u_k) - x_{k+1}) + \lambda_{N_r}^T r(x_0, x_K) \right)}_{\mathcal{L}(w, \lambda)}$$

The derivatives of the Lagrangian function with respect to the state variables  $x_k$

- For  $k = 0$ , we have

$$\nabla_{x_0} \mathcal{L}(w, \lambda) = \nabla_{x_0} L(x_0, u_0) + \frac{\partial f(x_0, u_0)^T}{\partial x_0} \lambda_1 + \frac{\partial r(x_0, x_K)^T}{\partial x_0} \lambda_{N_r}$$

- For  $k = 1, \dots, K-1$ , we have

$$\nabla_{x_k} \mathcal{L}(w, \lambda) = \nabla_{x_k} L(x_k, u_k) + \frac{\partial f(x_k, u_k)^T}{\partial x_k} \lambda_{k+1} - \lambda_k$$

- For  $k = K$ , we have

$$\nabla_{x_K} \mathcal{L}(w, \lambda) = \nabla_{x_K} E(x_N) - \lambda_K + \frac{\partial r(x_0, x_K)^T}{\partial x_K} \lambda_{N_r}$$

## Problem formulations | Simultaneous approach (cont.)

Consider the generic term  $\nabla_{x_k} \mathcal{L}(w, \lambda)$ ,

$$\nabla_{x_k} \mathcal{L}(w, \lambda) = \underbrace{\begin{bmatrix} \frac{\partial \mathcal{L}(w, \lambda)}{\partial x_k^{(1)}} \\ \frac{\partial \mathcal{L}(w, \lambda)}{\partial x_k^{(2)}} \\ \vdots \\ \frac{\partial \mathcal{L}(w, \lambda)}{\partial x_k^{(N_x)}} \end{bmatrix}}_{N_x \times 1}$$

## Problem formulations | Simultaneous approach (cont.)

## Formulations

Simultaneous  
approach

Sequential approach

Consider the derivative of the dynamics,

$$\frac{\partial f(x_k, u_k)}{\partial x_k}$$

Remember the dynamics,

$$f(x_k, u_k) = \begin{bmatrix} f_1(x_k^{(1)}, \dots, x_K^{(N_x)}, u_k) \\ \vdots \\ f_{n_x}(x_k^{(1)}, \dots, x_K^{(N_x)}, u_k) \\ \vdots \\ f_{N_x}(x_k^{(1)}, \dots, x_K^{(N_x)}, u_k) \end{bmatrix}$$

## Problem formulations | Simultaneous approach (cont.)

## Formulations

Simultaneous  
approach

Sequential approach

For the derivative of the dynamics, we have

$$\frac{\partial f(x_k^{(1)}, \dots, x_k^{(N_x)}, u_k)}{\partial x_k} = \begin{bmatrix} \frac{\partial f_1(x_k^{(1)}, \dots, x_k^{(N_x)}, u_k)}{\partial x_k} \\ \vdots \\ \frac{\partial f_{n_x}(x_k^{(1)}, \dots, x_k^{(N_x)}, u_k)}{\partial x_k} \\ \vdots \\ \frac{\partial f_{N_x}(x_k^{(1)}, \dots, x_k^{(N_x)}, u_k)}{\partial x_k} \end{bmatrix}$$

## Formulations

Simultaneous approach

Sequential approach

**Problem formulations | Simultaneous approach (cont.)**

In more detail, we have

$$\frac{\partial f(x_k, u_k)}{\partial x_k} = \underbrace{\begin{bmatrix} \frac{\partial f_1(x_k, u_k)}{\partial x_k^{(1)}} & \frac{\partial f_1(x_k, u_k)}{\partial x_k^{(2)}} & \dots & \frac{\partial f_1(x_k, u_k)}{\partial x_k^{(N_x)}} \\ \frac{\partial f_2(x_k, u_k)}{\partial x_k^{(1)}} & \frac{\partial f_2(x_k, u_k)}{\partial x_k^{(2)}} & \dots & \frac{\partial f_2(x_k, u_k)}{\partial x_k^{(N_x)}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{N_x}(x_k, u_k)}{\partial x_k^{(1)}} & \frac{\partial f_{N_x}(x_k, u_k)}{\partial x_k^{(2)}} & \dots & \frac{\partial f_{N_x}(x_k, u_k)}{\partial x_k^{(N_x)}} \end{bmatrix}}_{N_x \times N_x}$$

For the product with the equality multiplier, we get

$$\underbrace{\frac{\partial f(x_k, u_k)^T}{\partial x_k}}_{N_x \times N_x} \underbrace{\lambda_{k+1}}_{N_x \times 1}$$

## Formulations

Simultaneous approach

Sequential approach

**Problem formulations | Simultaneous approach (cont.)**

Consider the derivatives of the boundary conditions, we have the terms

$$\frac{\partial r(x_0, x_K)}{\partial x_0}$$

$$\frac{\partial r(x_0, x_K)}{\partial x_K}$$

Remember the boundary constraints

$$r(x_0, x_K) = \begin{bmatrix} x_0^{(1)} - \bar{x}_0^{(1)} \\ x_0^{(2)} - \bar{x}_0^{(2)} \\ \vdots \\ x_0^{(N_x)} - \bar{x}_0^{(N_x)} \\ \hline x_K^{(1)} - \bar{x}_K^{(1)} \\ x_K^{(2)} - \bar{x}_K^{(2)} \\ \vdots \\ x_K^{(N_x)} - \bar{x}_K^{(N_x)} \end{bmatrix}_{N_r \times 1}$$

Formulations  
Simultaneous approach  
Sequential approach

## Problem formulations | Simultaneous approach (cont.)

For the derivative of the boundary constraints with respect to  $x_0$ , we have

$$\frac{\partial r \left( x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}, x_K \right)}{\partial x_0} = \begin{bmatrix} \frac{\partial r_1 \left( x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}, x_K \right)}{\partial x_0} \\ \frac{\partial r_2 \left( x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}, x_K \right)}{\partial x_0} \\ \vdots \\ \frac{\partial r_{N_x} \left( x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}, x_K \right)}{\partial x_0} \\ \hline \frac{\partial r_{N_x+1} \left( x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}, x_K \right)}{\partial x_0} \\ \frac{\partial r_{N_x+2} \left( x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}, x_K \right)}{\partial x_0} \\ \vdots \\ \frac{\partial r_{2N_x} \left( x_0^{(1)}, x_0^{(2)}, \dots, x_0^{(N_x)}, x_K \right)}{\partial x_0} \end{bmatrix}$$

Formulations  
Simultaneous approach  
Sequential approach

## Problem formulations | Simultaneous approach (cont.)

In more detail, we have

$$\frac{\partial r(x_0, x_K)}{\partial x_0} = \underbrace{\begin{bmatrix} \frac{\partial r_1(x_0, x_K)}{\partial x_0^{(1)}} & \frac{\partial r_1(x_0, x_K)}{\partial x_0^{(2)}} & \dots & \frac{\partial r_1(x_0, x_K)}{\partial x_0^{(N_x)}} \\ \frac{\partial r_2(x_0, x_K)}{\partial x_0^{(1)}} & \frac{\partial r_2(x_0, x_K)}{\partial x_0^{(2)}} & \dots & \frac{\partial r_2(x_0, x_K)}{\partial x_0^{(N_x)}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial r_{2N_r}(x_0, x_K)}{\partial x_0^{(1)}} & \frac{\partial r_{2N_r}(x_0, x_K)}{\partial x_0^{(2)}} & \dots & \frac{\partial r_{2N_r}(x_0, x_K)}{\partial x_0^{(N_x)}} \end{bmatrix}}_{2N_r \times N_x}$$

For the product with the equality multiplier, we get

$$\underbrace{\frac{\partial r(x_0, x_K)^T}{\partial x_0}}_{\substack{N_x \times 2N_r \\ \hline N_x \times 1}} \underbrace{\lambda_{k+1}}_{2N_r \times 1}$$

## Problem formulations | Simultaneous approach (cont.)

Formulations

Simultaneous  
approach

Sequential approach

$$E(x_K) + \underbrace{\sum_{k=0}^{K-1} L(x_k, u_k) + \left( \sum_{k=0}^{K-1} \lambda_{k+1}^T (f(x_k, u_k) - x_{k+1}) + \lambda_{N_r}^T r(x_0, x_K) \right)}_{\mathcal{L}(w, \lambda)}$$

The derivatives of the Lagrangian function with respect to the control variables  $u_k$

- For  $k = 0, \dots, K - 1$ , we have

$$\nabla_{u_k} \mathcal{L}(w, \lambda) = \nabla_{u_k} L(x_k, u_k) + \frac{\partial f(x_k, u_k)^T}{\partial u_k} \lambda_{k+1}$$

## Problem formulations | Simultaneous approach (cont.)

Formulations

Simultaneous  
approach

Sequential approach

$$\begin{aligned}\nabla_w \mathcal{L}(w, \lambda) &= 0 \\ g(w) &= 0\end{aligned}$$

We can collect all the KKT conditions and solve them using a Newton-type method

- The approach solves the problem in the full space of the decision variables

## Formulations

Simultaneous  
approach

Sequential approach

**Problem formulations | Simultaneous approach (cont.)**

The approach can be extended to more general discrete-time optimal control problems

$$\begin{aligned} & \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} E(x_K) + \sum_{k=0}^{K-1} L_k(x_k, u_k) \\ \text{subject to } & x_{k+1} - f_k(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\ & h_k(x_k, u_k) \leq 0, \quad k = 0, 1, \dots, K-1 \\ & R_K(x_K) + \sum_{k=0}^{K-1} r_k(x_k, u_k) = 0 \\ & h_K(x_K) \leq 0 \end{aligned}$$

All problem functions are explicitly time-varying and we have also a terminal inequality

- Moreover, the boundary conditions are expressed in general form
- 

By collecting all variables in the vector  $w$ , we have the complete Lagrangian function

$$\mathcal{L}(w, \lambda, \mu) = f(w) + \lambda^T g(w) + \mu^T h(w)$$

## Formulations

Simultaneous  
approach

Sequential approach

**The sequential approach****Problem formulations**

Formulations  
Simultaneous approach  
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## Problem formulations | Sequential approach

$$\begin{aligned} \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ \text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\ & h(x_k, u_k) \leq 0, \quad k = 0, 1, \dots, K-1 \\ & r(x_0, x_N) = 0 \end{aligned}$$

The **sequential approach** solves the same problem in a reduced space of variables

The idea is to eliminate all the state variables  $x_1, x_2, \dots, x_K$  by a forward simulation

$$\begin{aligned} x_0 &= x_0 \\ x_1 &= f(x_0, u_0) \\ x_2 &= f(x_1, u_1) \\ &= f(f(x_0, u_0), u_1) \\ x_3 &= f(x_2, u_2) \\ &= f(f(f(x_0, u_0), u_1), u_2) \\ \dots &= \dots \end{aligned}$$

Formulations  
Simultaneous approach  
Sequential approach

## Problem formulations | Sequential approach (cont.)

We can express the states as function of the initial condition and previous controls

$$\begin{aligned} x_0 &= \underbrace{x_0}_{\bar{x}_0(x_0)} \\ x_1 &= \underbrace{f(x_0, u_0)}_{\bar{x}_1(x_0, u_0)} \\ x_2 &= f(x_1, u_1) \\ &= \underbrace{f(f(x_0, u_0), u_1)}_{\bar{x}_2(x_0, u_0, u_1)} \\ x_3 &= f(x_2, u_2) \\ &= \underbrace{f(f(f(x_0, u_0), u_1), u_2)}_{\bar{x}_3(x_0, u_0, u_1, u_2)} \\ \dots &= \dots \end{aligned}$$

More generally, the dependence is on all the control variables and the initial condition

$$\begin{aligned} \bar{x}_0(x_0, u_0, u_1, \dots, u_{K-1}) &= x_0 \\ \bar{x}_{k+1}(x_0, u_0, u_1, \dots, u_{K-1}) &= f(\bar{x}_k(x_0, u_0, u_1, \dots, u_{K-1}), u_k), \quad k = 0, 1, \dots, K-1 \end{aligned}$$

## Problem formulations | Sequential approach

Formulations

Simultaneous  
approach

Sequential approach

$$\begin{aligned} \min_{\substack{x_0, x_1, \dots, x_K \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(x_K) + \sum_{k=0}^{K-1} L(x_k, u_k) \\ \text{subject to} \quad & x_{k+1} - f(x_k, u_k | \theta_x) = 0, \quad k = 0, 1, \dots, K-1 \\ & h(x_k, u_k) \leq 0, \quad k = 0, 1, \dots, K-1 \\ & r(x_0, x_N) = 0 \end{aligned}$$

We can re-write the general discrete-time optimal control problem in reduced form

$$\begin{aligned} \min_{\substack{x_0 \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(\bar{x}_K(x_0, u_0, u_1, \dots, u_{K-1})) + \sum_{k=0}^{K-1} L(\bar{x}_k(x_0, u_0, u_1, \dots, u_{K-1}), u_k) \\ \text{subject to} \quad & h(\bar{x}_k(x_0, u_0, u_1, \dots, u_{K-1}), u_k) \leq 0, k = 0, 1, \dots, K-1 \\ & r(x_0, \bar{x}_N(x_0, u_0, u_1, \dots, u_{K-1})) = 0 \end{aligned}$$

## Problem formulations | Sequential approach (cont.)

Formulations

Simultaneous  
approach

Sequential approach

$$\begin{aligned} \min_{\substack{x_0 \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(\bar{x}_K(x_0, u_0, u_1, \dots, u_{K-1})) + \sum_{k=0}^{K-1} L(\bar{x}_k(x_0, u_0, u_1, \dots, u_{K-1}), u_k) \\ \text{subject to} \quad & h(\bar{x}_k(x_0, u_0, u_1, \dots, u_{K-1}), u_k) \leq 0, k = 0, 1, \dots, K-1 \\ & r(x_0, \bar{x}_N(x_0, u_0, u_1, \dots, u_{K-1})) = 0 \end{aligned}$$

The **objective function**, sum of **stage costs**  $L(\bar{x}_k, u_k)$  and a **terminal cost**  $E(\bar{x}_K)$

$$\underbrace{\sum_{k=0}^{K-1} L(\bar{x}_k, u_k) + E(\bar{x}_K)}_{f(w) \in \mathcal{R}}$$

That is,

$$L(x_0, u_0) + L(\bar{x}_1, u_1) + \dots + L(\bar{x}_{K-1}, u_{K-1}) + E(\bar{x}_K)$$

Stage cost is a (potentially nonlinear and time-varying) function of state and controls

The **decision variables**,  $K \times N_u$  **control** and  $N_x$  **state variables**

$$\underbrace{(x_0) \cup (u_0, u_1, \dots, u_{K-1})}_{w \in \mathcal{R}^{K \times N_u + N_x}}$$

## Problem formulations | Sequential approach (cont.)

## Formulations

Simultaneous  
approach

Sequential approach

$$\begin{aligned} \min_{\substack{x_0 \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(\bar{x}_K(x_0, u_0, u_1, \dots, u_{K-1})) + \sum_{k=0}^{K-1} L(\bar{x}_k(x_0, u_0, u_1, \dots, u_{K-1}), u_k) \\ \text{subject to} \quad & h(\bar{x}_k(x_0, u_0, u_1, \dots, u_{K-1}), u_k) \leq 0, k = 0, 1, \dots, K-1 \\ & r(x_0, \bar{x}_N(x_0, u_0, u_1, \dots, u_{K-1})) = 0 \end{aligned}$$

The **equality constraints**, the  $N_r$  **boundary conditions**

$$\underbrace{r(x_0, \bar{x}_K)}_{g(w) \in \mathcal{R}^{N_g}} = 0$$

The **inequality constraints**

$$\underbrace{h(\bar{x}_k, u_k) \leq 0}_{h(w) \in \mathcal{R}^{N_h}} \quad (k = 0, 1, \dots, K-1)$$

## Problem formulations | Sequential approach (cont.)

## Formulations

Simultaneous  
approach

Sequential approach

$$\begin{aligned} \min_{\substack{x_0 \\ u_0, u_1, \dots, u_{K-1}}} \quad & E(\bar{x}_K(x_0, u_0, u_1, \dots, u_{K-1})) + \sum_{k=0}^{K-1} L(\bar{x}_k(x_0, u_0, u_1, \dots, u_{K-1}), u_k) \\ \text{subject to} \quad & h(\bar{x}_k(x_0, u_0, u_1, \dots, u_{K-1}), u_k) \leq 0, k = 0, 1, \dots, K-1 \\ & r(x_0, \bar{x}_N(x_0, u_0, u_1, \dots, u_{K-1})) = 0 \end{aligned}$$

The Lagrangian function of the problem,

$$\mathcal{L}(w, \lambda, \mu) = f(w) + \lambda^T g(w) + \mu^T h(w)$$

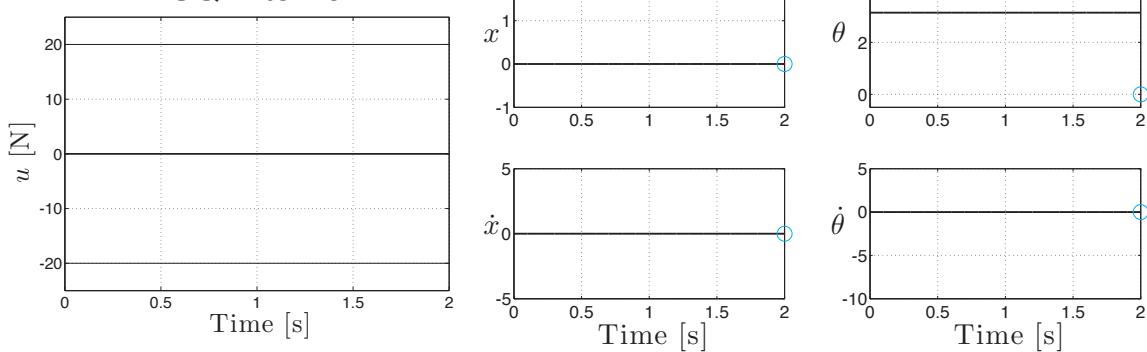
The Karush-Kuhn-Tucker conditions,

$$\begin{aligned} \nabla f(w^*) - \nabla g(w^*)\lambda^* - \nabla h(w^*)\mu^* &= 0 \\ g(w^*) &= 0 \\ h(w^*) &\geq 0 \\ \mu^* &\geq 0 \\ \mu_{n_h}^* h_{n_h}(w^*) &= 0, \quad n_h = 1, \dots, N_h \end{aligned}$$

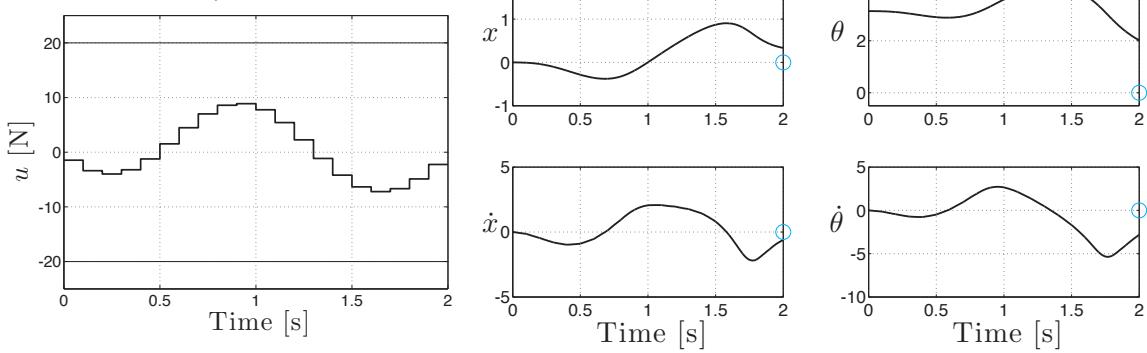
Formulations  
Simultaneous approach  
Sequential approach

## Problem formulations | Sequential approach (cont.)

SQP Iter: 0



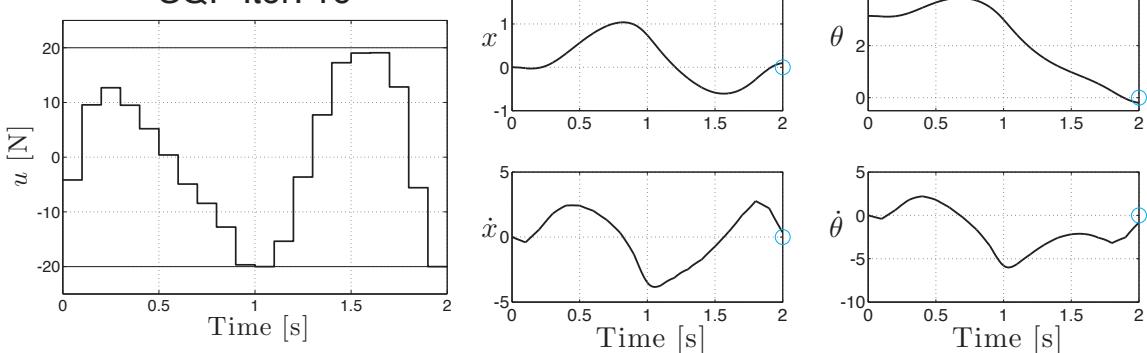
SQP Iter: 1



Formulations  
Simultaneous approach  
Sequential approach

## Problem formulations | Sequential approach (cont.)

SQP Iter: 16



For computational efficiency, it is preferable to use specific structure-exploiting solvers

- Such solvers recognise the sparsity properties of this class of problems