ELEC-C8201 Control and Automation
Lecture 2

Dynamic models, state-space representation, system responses

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Physical modelling

- Consider physical modelling of involved parameters in electrical circuits, mechanical systems (both linear and rotary motion) and flow systems.
- This review focuses on simple linear components, leaving, for example, thermal and energy considerations outside the scope of the review.

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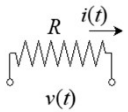
Basic components of electrical circuits

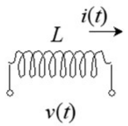
- Resistor (resistance)

$$v(t) = Ri(t)$$



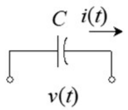
- Coil (inductance)

$$v(t) = L\frac{di(t)}{dt}$$



- Capacitor (capacitance)

$$i(t) = C\frac{dv(t)}{dt}$$



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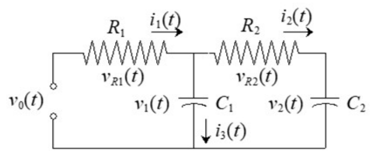
Example. Electrical circuit

- Making a model for the electrical circuit
  - The input, or impulse, is v0 (t) and the output quantities, i.e., the voltages V1 (t) and V2 (t).
  - Electric currents and resistors can be modelled as



$$C_1\frac{dv_1(t)}{dt} = i_3(t), \quad C_2\frac{dv_2(t)}{dt} = i_2(t) \qquad v_{R1}(t) = R_1 i_1(t), \quad v_{R2}(t) = R_2 i_2(t)$$

- Kirchoff's First Law
- Second Kirchoff law

$$i_1(t) = i_2(t) + i_3(t)$$

$$\begin{cases} v_0(t) = v_{R1}(t) + v_1(t) \\ v_1(t) = v_{R2}(t) + v_2(t) \end{cases} \Rightarrow \begin{cases} v_0(t) = R_1 i_1(t) + v_1(t) \\ v_1(t) = R_2 i_2(t) + v_2(t) \end{cases} \Rightarrow \begin{cases} i_1(t) = \dfrac{v_0(t) - v_1(t)}{R_1} \\ i_2(t) = \dfrac{v_1(t) - v_2(t)}{R_2} \end{cases}$$

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Example. Electrical circuit

- This model has voltages as states (memory elements), so it is advisable to eliminate the electrical currents as unnecessary variables from the developed equations

$$\begin{cases} \dfrac{dv_1(t)}{dt} = \dfrac{1}{C_1} i_3(t) = \dfrac{1}{C_1}\left(i_1(t) - i_2(t)\right) = \dfrac{v_0(t) - v_1(t)}{R_1 C_1} - \dfrac{v_1(t) - v_2(t)}{R_2 C_1} \\ \dfrac{dv_2(t)}{dt} = \dfrac{1}{C_2} i_2(t) = \dfrac{v_1(t) - v_2(t)}{R_2 C_2} \end{cases}$$

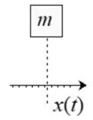
$$\Rightarrow \begin{cases} \dfrac{dv_1(t)}{dt} = -\left(\dfrac{1}{R_1 C_1} + \dfrac{1}{R_2 C_1}\right) v_1(t) + \left(\dfrac{1}{R_2 C_1}\right) v_2(t) + \left(\dfrac{1}{R_1 C_1}\right) v_0(t) \\ \dfrac{dv_2(t)}{dt} = \left(\dfrac{1}{R_2 C_2}\right) v_1(t) - \left(\dfrac{1}{R_2 C_2}\right) v_2(t) \end{cases}$$

Basic components of mechanical systems

Linear Motion:

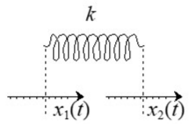
- Mass (inertia)

$$F_m(t) = m\frac{d^2x(t)}{dt^2}$$



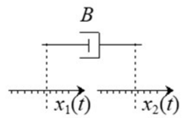
- Spring

$$F_k(t) = k\Delta x(t) = k(x_1(t) - x_2(t))$$



- Damper

$$F_b(t) = B\frac{d\Delta x(t)}{dt} = B\left(\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt}\right)$$



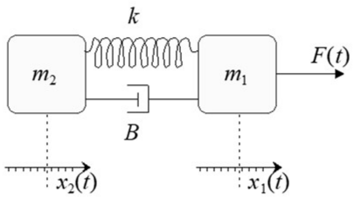
Example. Mechanical system

- Make a model for a mechanical system in which two mass pieces are connected together with a spring and a damper
- The input is an external force F (T) and the position of the second mass $x_2(t)$ is the output
- First mass force equation:

$$m_1(t)\frac{d^2x_1(t)}{dt^2} + B\left(\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt}\right) + k\left(x_1(t) - x_2(t)\right) = F(t)$$

- Second mass force equation:

$$m_2(t)\frac{d^2x_2(t)}{dt^2} = B\left(\frac{dx_1(t)}{dt} - \frac{dx_2(t)}{dt}\right) + k\left(x_1(t) - x_2(t)\right)$$

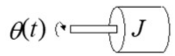


Basic components of mechanical systems

Rotational motion:

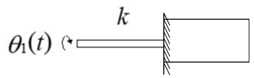
- Moment of inertia

$$T_J(t) = J\frac{d^2\theta(t)}{dt^2}$$



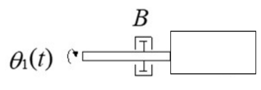
- Torque Spring

$$T_k(t) = k\theta_1(t)$$



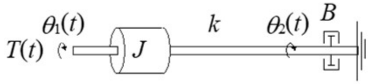
- Torque Damper

$$T_b(t) = B\frac{d\theta_1(t)}{dt}$$



Example. Mechanical system

- Make a model for the rotating system shown in the figure. The excitation is torque T (t) and the response is angles $\theta_1(t)$ and $\theta_2(t)$



$$\begin{cases} J\dfrac{d^2\theta_1(t)}{dt^2} + k\left(\theta_1(t) - \theta_2(t)\right) = T(t) \\ B\dfrac{d\theta_2(t)}{dt} = k\left(\theta_1(t) - \theta_2(t)\right) \end{cases}$$

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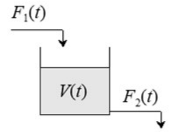
Basic components of flow systems

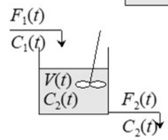
- Flow tank

$$\frac{dV(t)}{dt} = F_1(t) - F_2(t)$$



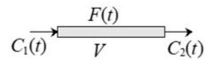
- Ideal Mixer

$$\frac{dV(t)C_2(t)}{dt} = F_1(t)C_1(t) - F_2(t)C_2(t)$$



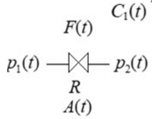
- Pipe flow delay

$$C_2(t) = C_1(t - T_d(t)) = C_1\left(t - \frac{V}{F(t)}\right)$$



- Flow through an orifice

$$F(t) = A(t)R\sqrt{\Delta p(t)} = A(t)R\sqrt{p_1(t) - p_2(t)}$$



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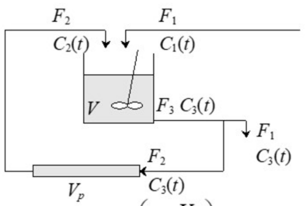
Example. Flow System

- The flow system is shown in the diagram. There is the input flow concentration C1 (t) and the concentration of the output flow is C3(t). Flow and volumes are constants
- The flow branching point can be

$$F_3 = F_1 + F_2$$



- Considering ideal sensor and pipe flow

$$\frac{dVC_3(t)}{dt} = F_1C_1(t) + F_2C_2(t) - F_3C_3(t), \quad C_2(t) = C_3\left(t - \frac{V_p}{F_2}\right)$$

- Eliminating the variable $C_2(t)$ and F_3:

$$\frac{dC_3(t)}{dt} = \frac{1}{V}\left(F_1C_1(t) + F_2C_3\left(t - \frac{V_p}{F_2}\right) - (F_1 + F_2)C_3(t)\right)$$

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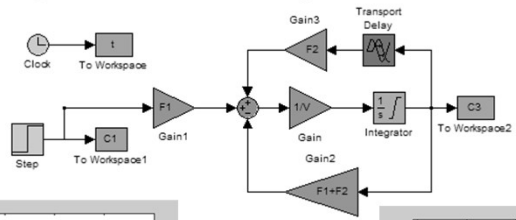
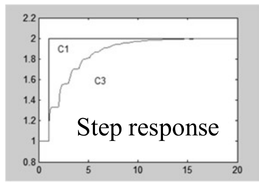
Example. Flow System Simulation

- Configuring simulation model for flow system with Simullink



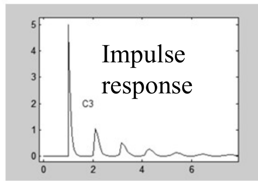


$V = 0.2$

$V_p = 2$

$F_1 = 1$

$F_2 = 2$



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State Space Representation

- The state space representation is a compact way of representing high-order differential equations/systems.
- The instantaneous state of the system is a complete description of the system. If the initial state (state quantities at the beginning) and all the input quantities are known from the beginning, then the system state and the output quantities can be determined at an arbitrary time. It follows that the state space representation is very suitable for simulation.
- The control of state variables allows for better system control compared with the control of the system's output quantities.
- The state space representation is a standard-format representation, so the management mechanisms can be standardized independently of the system (equations are independent of the individual system number and parameters)
- The state space representation is suitable for modeling and managing multivariate systems

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State Space Representation

- In a state space representation, an arbitrary order differential equation/system is represented as a group of first-order differential equations.
- The selection of spaces can be made in infinitely different ways = > The state space representation is not unique but many different state variables can describe the same input/output model.
- The general state space representation is of the format $\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases}$
- $\mathbf{x}(t)$ is the state, $\mathbf{u}(t)$ the control input $\mathbf{y}(t)$ output - All these quantities can be vectors or scalars.
- $\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t))$ is the system equation (describing system dynamics) and $\mathbf{g}(\mathbf{x}(t),\mathbf{u}(t))$ is the starting description (explains how the output quantities depend on the input and the state)
- If $\mathbf{u}(t)$ is a scalar $u(t)$ and $\mathbf{y}(t)$ is a scalar $y(t)$, then this is a SISO system-regardless of the vector x(t)'s dimension.

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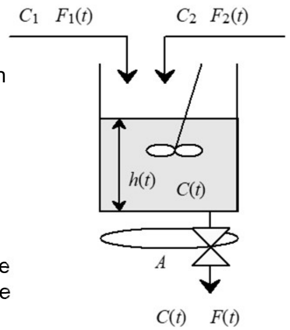
Example. Flow System in State Space representation

- The flow process is mixed with antifreeze (dilute solution with a chemical concentration of C1 in a spring solution with a concentration of C2).
- The objective is to obtain the desired production volume (flow F) of the product (with concentration C) that meets the specifications, using the flow rate (F1 and F2).
- The mixing tank is connected to a discharge valve open to atmospheric pressure = > The removal flow is proportional to the square root of the surface:
- $$F(t) = k\sqrt{h(t)}$$



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Example. Flow System in State Space representation

- A balance equation is formed (simplified volume balance) and component balance

$$\begin{cases} \dfrac{dV(t)}{dt} = A\dfrac{dh(t)}{dt} = F_1(t) + F_2(t) - F(t) \\ \dfrac{dC(t)V(t)}{dt} = C_1F_1(t) + C_2F_2(t) - C(t)F(t) \end{cases}$$

$$\frac{dC(t)V(t)}{dt} = \frac{dC(t)}{dt}V(t) + C(t)\frac{dV(t)}{dt} = \frac{dC(t)}{dt}V(t) + C(t)\big(F_1(t) + F_2(t) - F(t)\big)$$

$$\Rightarrow \begin{cases} A\dfrac{dh(t)}{dt} = F_1(t) + F_2(t) - k\sqrt{h(t)} \\ \dfrac{dC(t)}{dt}Ah(t) = \big(C_1 - C(t)\big)F_1(t) + \big(C_2 - C(t)\big)F_2(t) \end{cases}$$

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Example. Flow System in State Space representation

- Obtain a simple equation group of first-order differential equations

$$\Rightarrow \begin{cases} \dfrac{dh(t)}{dt} = \dfrac{1}{A}\left(F_1(t)+F_2(t)-k\sqrt{h(t)}\right) \\ \dfrac{dC(t)}{dt} = \dfrac{1}{Ah(t)}\left((C_1-C(t))F_1(t)+(C_2-C(t))F_2(t)\right) \end{cases}$$

- Choose H and C for state , F1 and F2 for input and output quantities as F and C

$$\mathbf{x}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}=\begin{bmatrix} h(t) \\ C(t) \end{bmatrix},\quad \mathbf{u}(t)=\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}=\begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix},\quad \mathbf{y}(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}=\begin{bmatrix} F(t) \\ C(t) \end{bmatrix}$$

- These variable selections can be used to write directly in the standard format:

$$\begin{cases} \dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t)) \\ \mathbf{y}(t)=\mathbf{g}(\mathbf{x}(t),\mathbf{u}(t)) \end{cases}$$

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Example. Flow System in State Space representation

- Writing the system equation:

$$\dot{\mathbf{x}}(t)=\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}=\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t))=\begin{bmatrix} f_1(\mathbf{x}(t),\mathbf{u}(t)) \\ f_2(\mathbf{x}(t),\mathbf{u}(t)) \end{bmatrix}=\begin{bmatrix} \dfrac{1}{A}\left(u_1(t)+u_2(t)-k\sqrt{x_1(t)}\right) \\ \dfrac{1}{Ax_1(t)}\left((C_1-x_2(t))u_1(t)+(C_2-x_2(t))u_2(t)\right) \end{bmatrix}$$

- For the initial description, the dependency of the output variables on state variables

$$\mathbf{y}(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}=\begin{bmatrix} F(t) \\ C(t) \end{bmatrix}=\begin{bmatrix} k\sqrt{h(t)} \\ C(t) \end{bmatrix}=\begin{bmatrix} k\sqrt{x_1(t)} \\ x_2(t) \end{bmatrix}$$

- Initial description:

$$\mathbf{y}(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}=\mathbf{g}(\mathbf{x}(t),\mathbf{u}(t))=\begin{bmatrix} g_1(\mathbf{x}(t),\mathbf{u}(t)) \\ g_2(\mathbf{x}(t),\mathbf{u}(t)) \end{bmatrix}=\begin{bmatrix} k\sqrt{x_1(t)} \\ x_2(t) \end{bmatrix}$$

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Example. Flow System in State Space representation

- The flow process state space representation can be

$$\begin{cases} \dot{\mathbf{x}}(t)=\begin{bmatrix} \dfrac{1}{A}\left(u_1(t)+u_2(t)-k\sqrt{x_1(t)}\right) \\ \dfrac{1}{Ax_1(t)}\left((C_1-x_2(t))u_1(t)+(C_2-x_2(t))u_2(t)\right) \end{bmatrix} \\ \mathbf{y}(t)=\begin{bmatrix} k\sqrt{x_1(t)} \\ x_2(t) \end{bmatrix} \end{cases}$$

- In this example, the selection of modes was easy because the appropriate state variables were obtained directly from the system model. Examining other methods for selecting variables in connection with linear mode representation.

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Linear mode representation

- The previous example described a non-linear state space representation. If the system being viewed is linear, its variables and parameters can be assembled into separate vectors and matrix to obtain a standard linear mode representation.

$$\begin{cases} \dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t),\mathbf{u}(t))=\mathbf{A}\mathbf{x}(t)+\mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t)=\mathbf{g}(\mathbf{x}(t),\mathbf{u}(t))=\mathbf{C}\mathbf{x}(t)+\mathbf{D}\mathbf{u}(t) \end{cases}$$

- The parameter array **A** is called system matrix, **B** control matrix, **C** output matrix, and **D** as a direct effect matrix. Often D = 0, in which case the entire direct effect term disappears from the state representation. (This is done with a strictly proper system).

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Linear mode representation

- Differential equations

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + \cdots + a_{1n}x_n + b_{11}u_1 + \cdots + b_{1m}u_m \\ \dot{x}_2 = a_{21}x_1 + \cdots + a_{2n}x_n + b_{21}u_1 + \cdots + b_{2m}u_m \\ \vdots \\ \dot{x}_n = a_{n1}x_1 + \cdots + a_{nn}x_n + b_{n1}u_1 + \cdots + b_{nm}u_m \end{cases}$$

- Can be presented as an array equation

$$\begin{bmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

$$\text{eli:} \qquad \dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu}$$

Electrical Circuit State Space representation

- Examine the electrical circuits of previous example and develop the state

$$\begin{cases} \dfrac{dv_1(t)}{dt} = -\left(\dfrac{1}{R_1C_1} + \dfrac{1}{R_2C_1}\right)v_1(t) + \left(\dfrac{1}{R_2C_1}\right)v_2(t) + \left(\dfrac{1}{R_1C_1}\right)v_0(t) \\ \dfrac{dv_2(t)}{dt} = \left(\dfrac{1}{R_2C_2}\right)v_1(t) - \left(\dfrac{1}{R_2C_2}\right)v_2(t) \end{cases}$$

- The natural selection of state variables is the voltages of the capacitors (because they are represented with first order differential equations). Only the voltage of the second capacitor is now selected as the output v_2.

$$\mathbf{x}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}, \quad u(t) = v_0(t), \quad y(t) = v_2(t)$$

- These choices provide

State-space representation of an electrical circuit

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} -\left(\dfrac{1}{R_1C_1} + \dfrac{1}{R_2C_1}\right)x_1(t) + \dfrac{1}{R_2C_1}x_2(t) + \dfrac{1}{R_1C_1}u(t) \\ \dfrac{1}{R_2C_2}x_1(t) - \dfrac{1}{R_2C_2}x_2(t) \end{bmatrix} = \mathbf{f}(\mathbf{x}(t), u(t)) \\ y(t) = x_2(t) = g(\mathbf{x}(t), u(t)) \end{cases}$$

- And thus:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} -\left(\dfrac{1}{R_1C_1} + \dfrac{1}{R_2C_1}\right) & \dfrac{1}{R_2C_1} \\ \dfrac{1}{R_2C_2} & -\dfrac{1}{R_2C_2} \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \dfrac{1}{R_1C_1} \\ 0 \end{bmatrix} u(t) = \mathbf{Ax}(t) + \mathbf{B}u(t) \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x}(t) + 0\,u(t) = \mathbf{Cx}(t) + Du(t) \end{cases}$$

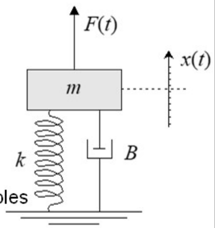
Forming a State Space representation

- How to generate a state space representation systematically?
  - Choosing physical and rational state variables from model equations (as in previous examples)
  - Using a derivative operator p
  - Using canonical forms
- Physically correct state variables
  - Often the easiest way
- Using the "derivatives" operator
  - can be formed e.g. Canonical shape, or in some cases a diagonal shape, should be controlled or observed

Forming a State Space representation

- Examining the mechanical System

$$m\ddot{x}(t)+B\dot{x}(t)+kx(t)=F(t)$$



- Physically relevant state variables
  - Select position x and speed $v = dx/dt$ $\quad \begin{cases} x_1(t)=x(t) \\ x_2(t)=\dot{x}(t)\end{cases}$

$$y(t)=x(t)=x_1(t),\quad u(t)=F(t)$$

  - Determining the derivatives of the selected state variables

$$\begin{cases}\dot{x}_1(t)=\dot{x}(t)=x_2(t)\\ \dot{x}_2(t)=\ddot{x}(t)=-\frac{B}{m}\dot{x}(t)-\frac{k}{m}x(t)+\frac{1}{m}F(t)=-\frac{B}{m}x_2-\frac{k}{m}x_1(t)+\frac{1}{m}u(t)\end{cases}$$

$$\begin{cases}\dot{\mathbf{x}}(t)=\begin{bmatrix}0 & 1\\ -\frac{k}{m} & -\frac{B}{m}\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}u(t)\\ y(t)=\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{x}(t)\end{cases}$$

Forming a State Space representation

- Second method: Use the *p operator* $p = d/dt$
  - You write the original equation using the p operator whenever there is a differentiation:

$$\ddot{x}(t)+\frac{B}{m}\dot{x}(t)=-\frac{k}{m}x(t)+\frac{1}{m}F(t)\ \Rightarrow\ p\{\overbrace{p\{\overbrace{x(t)}^{x_1(t)}\}+\frac{B}{m}x(t)}^{x_2(t)}\}=-\frac{k}{m}x(t)+\frac{1}{m}F(t)$$

$$\begin{cases}x_1(t)=x(t)\\ x_2(t)=p\{x(t)\}+\frac{B}{m}x(t)=\dot{x}_1(t)+\frac{B}{m}x_1(t)\\ p\{x_2(t)\}=\dot{x}_2(t)=-\frac{k}{m}x(t)+\frac{1}{m}F(t)=-\frac{k}{m}x_1(t)+\frac{1}{m}u(t)\end{cases}$$

$$\begin{cases}\dot{x}_1(t)=-\frac{B}{m}x_1(t)+x_2(t)\\ \dot{x}_2(t)=-\frac{k}{m}x_1(t)+\frac{1}{m}u(t)\\ y(t)=x(t)=x_1(t)\end{cases}$$

$$\begin{cases}\dot{\mathbf{x}}(t)=\begin{bmatrix}-\frac{B}{m} & 1\\ -\frac{k}{m} & 0\end{bmatrix}\mathbf{x}(t)+\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}u(t)\\ y(t)=\begin{bmatrix}1 & 0\end{bmatrix}\mathbf{x}(t)\end{cases}$$

Differential equation to transfer function

The differential equation can also be used as a shortcut to the transfer function. Common linear differential equation

$$y^{(n)}(t)+a_1y^{(n-1)}(t)+\cdots+a_{n-1}y^{(1)}(t)+a_ny(t)=b_1u^{(n-1)}(t)+\cdots+b_{n-1}u^{(1)}(t)+b_nu(t)$$

is Laplace-transformed (and assuming initial values as zeros)

$$\left(s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n\right)Y(s)=\left(b_1s^{n-1}+\cdots+b_{n-1}s+b_n\right)U(s)$$

This makes it easy to create a transfer function

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_1s^{n-1}+\cdots+b_{n-1}s+b_n}{s^n+a_1s^{n-1}+\cdots+a_{n-1}s+a_n}$$

Similarly, the transfer function can be re-transformed to differential equations

Example: Mass block

Specify the transfer function and weighting function in the previous example

$$\ddot{x}(t)+2\dot{x}(t)+5x(t)=F(t)\qquad \begin{cases}y(t)=x(t)\\ u(t)=F(t)\end{cases}$$

$$\ddot{y}(t)+2\dot{y}(t)+5y(t)=u(t)\ \Rightarrow\ s^2Y(s)+2sY(s)+5Y(s)=U(s)$$

$$\left(s^2+2s+5\right)Y(s)=U(s)\ \Rightarrow\ G(s)=\frac{Y(s)}{U(s)}=\frac{1}{s^2+2s+5}$$

$$g(t)=L^{-1}\{G(s)\}=L^{-1}\left\{\frac{1}{s^2+2s+5}\right\}=\frac{1}{2}L^{-1}\left\{\frac{2}{(s+1)^2+2^2}\right\}=\frac{1}{2}e^{-t}\sin(2t)$$

The weighting function is now the same as the unit impulse response

Determination of response

When the transfer function is known, the response (forced response) is calculated as follows

- Laplace conversion of external control $u(t)$ $\quad U(s) = L\{u(t)\}$
- Solve for output $Y(s)$ in Laplace domain $\quad Y(s) = G(s) \cdot U(s)$
- Take inverse Laplace transform of output $\quad y(t) = L^{-1}\{Y(s)\}$

Or: $y(t) = L^{-1}\{G(s) \cdot L\{u(t)\}\}$

Example: Mass block

Determine the unit step and ramp responses for the mass position. A transfer function was previously assigned to the system as: $G(s) = \frac{1}{s^2+2s+5}$

Unit step: $U(s) = \frac{1}{s}$

Response: $Y(s) = G(s) \cdot U(s) = \frac{1}{s^2+2s+5} \cdot \frac{1}{s} = \frac{1}{s(s^2+2s+5)}$

Taking partial fractions:

$$\frac{1}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5} = \frac{A(s^2+2s+5)+s(Bs+C)}{s(s^2+2s+5)} = \frac{(A+B)s^2+(2A+C)s+5A}{s(s^2+2s+5)}$$

$$\Rightarrow \quad (A+B)s^2+(2A+C)s+5A \equiv 1 \quad \Rightarrow \begin{cases} A+B=0 \\ 2A+C=0 \\ 5A=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{5} \\ B=-\frac{1}{5} \\ C=-\frac{2}{5} \end{cases}$$

Example: Mass block

$$Y(s) = \frac{1}{s(s^2+2s+5)} = \frac{\frac{1}{5}}{s} - \frac{\frac{1}{5}s+\frac{2}{5}}{s^2+2s+5} = \frac{1}{5}\left(\frac{1}{s} - \left(\frac{s+1}{(s+1)^2+2^2} + \frac{1}{2}\frac{2}{(s+1)^2+2^2}\right)\right)$$

The response can be:

$$y(t) = L^{-1}\{Y(s)\} = \tfrac{1}{5}\left(1-\left(e^{-t}\cos(2t)+\tfrac{1}{2}e^{-t}\sin(2t)\right)\right)$$

Ramp input:

$U(s) = \frac{1}{s^2} \qquad Y(s) = G(s) \cdot U(s) = \frac{1}{s^2+2s+5} \cdot \frac{1}{s^2} = \frac{1}{s^2(s^2+2s+5)}$

As before, taking partial fractions and solving:

$$y(t) = L^{-1}\{Y(s)\} = \tfrac{1}{5}\left(t-\tfrac{2}{5}+\tfrac{2}{5}e^{-t}\cos(2t)-\tfrac{3}{10}e^{-t}\sin(2t)\right)$$

Steady State Response

Steady state response tells you how much the signal is strengthened or dampened after passing through the system

- With the unit step response, steady state amplification indicates to which level the response will remain (asymptotically stable system)
- With the ramp response, the amplification indicates the continuity of the slope of the response (Asymptotically stable system)

Steady state response can be calculated from the transfer function using the limit value theorem. Steady state response can also be set for unstable system transfer functions, but it does not have a physical interpretation that is linked to the end value of the response.

Steady state response (static gain) is: $\bar{k} = \lim_{s \to 0}\{G(s)\}$

Example: Mass block

Evaluating the mass block system for Steady State response:

$$G(s) = \frac{1}{s^2 + 2s + 5}$$

$$\Rightarrow \quad \bar{k} = \lim_{s \to 0}\{G(s)\} = \lim_{s \to 0}\left\{\frac{1}{s^2 + 2s + 5}\right\} = \frac{1}{5}$$

Also seen in time domain as t tends to ∞

- Unit Step Response $y(t) = \frac{1}{5}\left(1 - \left(e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)\right)\right)$
- Ramp response $y(t) = \frac{1}{5}\left(t - \frac{2}{5} + \frac{2}{5}e^{-t}\cos(2t) - \frac{3}{10}e^{-t}\sin(2t)\right)$

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MATLAB: Models, Inputs and Responses

- Enter a template function in the workspace:
  - One can write in transfer function or state space **`tf`**, **`zpk`** & **`ss`** –commands.
- For example, consider the mechanical system in example 2:
  - The system depicts a differential equation:

$$m\ddot{x}(t) + B\dot{x}(t) + kx(t) = F(t) \quad \Rightarrow \quad \ddot{x}(t) + 2\dot{x}(t) + 5x(t) = u(t)$$

  - Entering in workspace as: **`sys=tf(1,[1 2 5])`**

```
Transfer function:        1
                    -------------
                    s^2 + 2 s + 5
```

  - Examining Impulse and step responses

```
[imp1,t1]=impulse(sys);
(Tai:   [imp1,t1]=impulse(tf(1,[1 2 5])); )
[ste2,t2]=step(sys);
plot(t1,imp1)
plot(t2,ste2)
```

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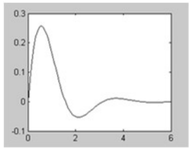
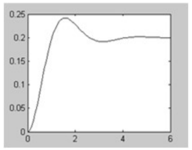
MATLAB: Models, Inputs and Responses

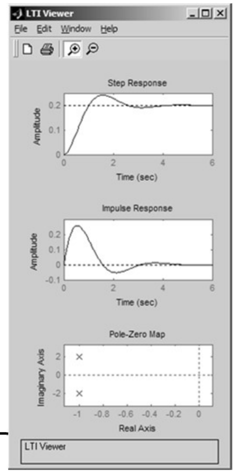
- Output responses:





- For confirmation:
  - **`k=dcgain(sys)`**

    `k =0.2000`
- Responses can also be examined in **`ltiview`**-LTI window
  - **`ltiview`**
    - **`file -> import -> sys`**
    - **`edit -> plot configurations`**



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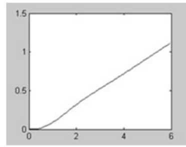
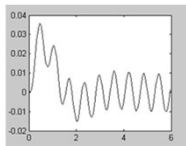
MATLAB: Models, Inputs and Responses

- In addition to the basic responses, the MATLAB's workspace allows you to calculate responses to arbitrary control inputs using the lsim command.
  - Ramp response
    - **`ram=lsim(sys,t2,t2);`**
    - **`plot(t2,ram)`**
  - Sine response
    - **`osc=lsim(sys,sin(10*t2),t2);`**
    - **`plot(t2,osc)`**
  - **`lsim`**–For a command, you can use some other command to take a calculated time vector, or generate one yourself, either by using a generic Matlab command (colons) or by using a special linspace command
    - **`t3=(0:0.1:10)';`**
    - **`t3=linspace(0,10,101)';`**





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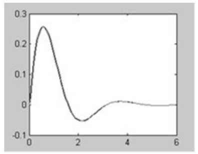
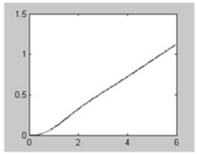
MATLAB: Models, Inputs and Responses

- In earlier lectures, it was told that the basic responses were derived from each other by either derivation or integration. Now, we see how well this works numerically
- Take the base response as step function and find out impluse response by derivation and slope (ramp) response by integration:
  - The numerical derivative of a step response:
  - `imp2=diff(ste2)./diff(t2);`
    - `plot(t2,[0;imp2],t1,imp1)`
  - Step Response Numerical Integral:
  - `delta=mean(diff(t2))`
    - `ram2=cumsum(ste2)*delta;`
    - `plot(t2,[ram2 ram])`
- Numerical derivatives and integration work reasonably well in this (undisturbed) case.





From State Space to Transfer Function form

The solution of convolution integration is laborious and often easier to achieve if the state equation is left on Laplace form, with the time domain function just written as Laplace inverse form (just as in the overall response)

$$\mathbf{X}(s) = (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)$$

$$\mathbf{x}(t) = L^{-1}\left\{(s\mathbf{I}-\mathbf{A})^{-1}\right\}\mathbf{x}(0) + L^{-1}\left\{(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)\right\}$$

$$= \mathbf{\Phi}(t)\mathbf{x}(0) + L^{-1}\left\{(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)\right\} = \mathbf{x_0}(t) + \mathbf{x_u}(t)$$

The output is obtained accordingly:

$$\mathbf{Y}(s) = \mathbf{C}\mathbf{X}(s) + \mathbf{D}\mathbf{U}(s) = \mathbf{C}\left((s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s)\right) + \mathbf{D}\mathbf{U}(s)$$

$$= \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{x}(0) + \left(\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}\right)\mathbf{U}(s) = \mathbf{C}\mathbf{\Phi}(s)\mathbf{x}(0) + \mathbf{G}(s)\mathbf{U}(s)$$

$$= \mathbf{Y_0}(s) + \mathbf{Y_u}(s)$$

$$\mathbf{y}(t) = L^{-1}\left\{\mathbf{Y}(s)\right\}$$

From State Space to Transfer Function form

Given below is a formula to define a transfer function from a state space representation (the transfer function was defined as dependent on the input quantities and outputs in the Laplace plane – when the initial values are not taken into consideration)

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Often the feedforward term D is zero, in which case the formula is even simpler to form:

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$$

Example: Mass block

- Solving a mass block Unit step response (forced response) using the transfer function

$$G(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}s & -1\\ 5 & s+2\end{bmatrix}^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}$$

$$= \frac{\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}s+2 & 1\\ -5 & s\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}}{s^2+2s+5} = \frac{1}{s^2+2s+5}$$

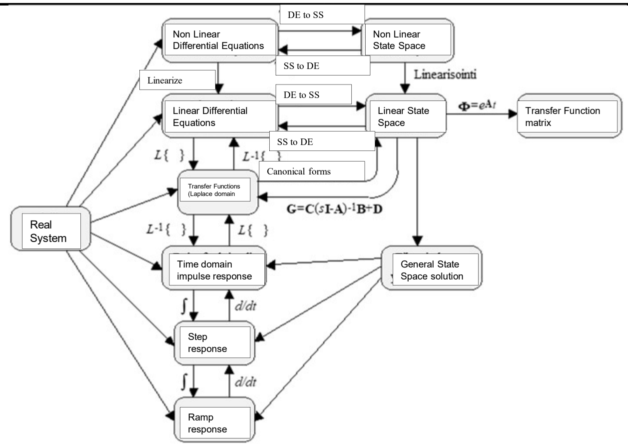
- The response

$$U(s) = \frac{1}{s}$$

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\{G(s)U(s)\} = L^{-1}\left\{\frac{1}{s(s^2+2s+5)}\right\}$$

$$= \tfrac{1}{5}\left(1 - \left(e^{-t}\cos(2t) + \tfrac{1}{2}e^{-t}\sin(2t)\right)\right)$$

Models and conversions between them:



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