1. It must be demonstrated that the differential equation below and the state presentation are identical.

$$\ddot{y}(t) + 4\dot{y}(t) + 3y(t) = 2u(t) \Leftrightarrow \begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

Using matrix multiplication rules, the equation group reduces to the following matrix form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 \cdot x_1(t) + 1 \cdot x_2(t) \\ -3 \cdot x_1(t) - 4 \cdot x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \cdot u(t) \\ 2 \cdot u(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -3x_1(t) - 4x_2(t) + 2u(t) \end{bmatrix}$$

$$y(t) = 1 \cdot x_1(t) + 0 \cdot x_2(t) = x_1(t)$$

$$\Rightarrow \begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -3x_1(t) - 4x_2(t) + 2u(t) \\ y(t) = x_1(t) \end{cases}$$

From these equations, we find the IO (input-output) $u(t)$ and $y(t)$ interdependence. Now we need to eliminate the state variables $x_1(t)$, and $x_2(t)$ from the final equation. Initially, using the last equation $y(t) = x_1(t)$ relation into the previous two:

$$\begin{cases} \dot{y}(t) = x_2(t) \\ \dot{x}_2(t) = -3y(t) - 4x_2(t) + 2u(t) \end{cases}$$

Now place it in the first equation $\dot{y}(t) = x_2(t)$:

$$\ddot{y}(t) = -3y(t) - 4\dot{y}(t) + 2u(t) \Rightarrow \ddot{y}(t) + 4\dot{y}(t) + 3y(t) = 2u(t)$$

This shows that the differential equation and the state-space equation describe the same system.

2. First write the same equation a little differently by using an entry for the derivative given by:

$$p = \frac{d}{dt}, \quad p^2 = \frac{d^2}{dt^2}, \quad \cdots$$

$$\ddot{y}(t) - 3\dot{y}(t) + 2y(t) = 2\dot{u}(t)$$

$$\Leftrightarrow \frac{d^2}{dt^2} y(t) - 3\frac{d}{dt} y(t) + 2y(t) = 2\frac{d}{dt} u(t)$$

$$\Leftrightarrow p^2 y(t) - 3py(t) + 2y(t) = 2pu(t)$$

Transfer all the terms containing the p, or the derivative, to the left and all the others to the right

$$\Leftrightarrow p^2 y(t) - 3py(t) - 2pu(t) = -2y(t)$$

Takes p as the common factor. (**Note**: keep taking p as a common factor till no higher order of p remains in the equation**.)**

$$\Leftrightarrow p\big(py(t) - 3y(t) - 2u(t)\big) = -2y(t)$$

Now select the state variables as the bracketed values which are multipled by each p.

$$p\bigg(\underbrace{\underbrace{p\,y(t)}_{x_1} - 3y(t) - 2u(t)}_{x_2} \bigg) = -2y(t) \tag{1}$$

Or:

$$\begin{cases} x_1(t) = y(t) \\ x_2(t) = \dot{x}_1(t) - 3x_1(t) - 2u(t) \end{cases}$$

The second equation is placed first and can be solved $\dot{x}_1(t)$ (below). Expressions $\dot{x}_2(t)$ is obtained from the equation (1).

$$\begin{cases} \dot{x}_1(t) = 3x_1(t) + x_2(t) + 2u(t) \\ \dot{x}_2(t) = -2x_1(t) \end{cases}$$

The output y is described as:

$$y(t) = x_1(t)$$

Based on the last three equations, the following state-space representation can be written

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

The state space variables can be selected in multiple ways, as long as the result is in an acceptable form. In fact, state variables can be selected in infinite many ways. All of them lead to the same input-output behavior. (Of course, the system will not be changed based on the state variable selection.) In this task, the selection of state variables, unlike the first problem, can be challenging, as the derivative on the right side of the differential equation is problematic. There must be no derivative terms on the right side of equations.

3. *RLC Circuits* - The modelling of circuits is done through the basic formulae of electrical engineering (Kirchhoff I & II...). The equations describing the components needed for this task are:

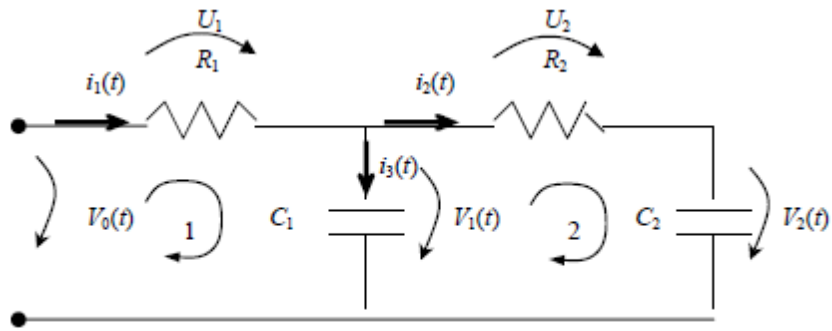   **Resistor**

   $U = RI$

   Capacitor

   $I = C \dfrac{dU}{dt}$

The purpose is to form the differential equation, which shows the interdependence between control input and output. In the given task, the response should be of the form:

$$V_2^{(n)}(t) + a_1 V_2^{(n-1)}(t) + \cdots a_{n-1} V_2^{(1)}(t) + a_n V_2(t) = b_1 V_0^{(n-1)}(t) + \cdots b_{n-1} V_0^{(1)}(t) + b_n V_0(t)$$

Forming the required loops and voltage balance based on the Kirchhoff law:

For ease of writing and reading, we omit tagging time dependency. Voltage balance across resistors:

$$U_1 = R_1 i_1 \text{ and } U_2 = R_2 i_2$$

Current flow through capacitors:

$$i_2 = C_2 \dot{V}_2 \text{ and } i_3 = C_1 \dot{V}_1$$

Kirchoff's first law:

$$i_1 = i_2 + i_3 = C_1 \dot{V}_1 + C_2 \dot{V}_2$$

Loop 1: $\quad V_0 = R_1 i_1 + V_1$
Loop 2: $\quad V_1 = R_2 i_2 + V_2$

Substituting for current values:

$$\begin{cases} V_0 = R_1 (C_1 \dot{V}_1 + C_2 \dot{V}_2) + V_1 \\ V_1 = R_2 C_2 \dot{V}_2 + V_2 \end{cases}$$

We make the final differential equation by eliminating V_1 from the above equation as follows:

$$V_1 = R_2 C_2 \dot{V}_2 + V_2$$
$$\dot{V}_1 = R_2 C_2 \ddot{V}_2 + \dot{V}_2$$

Substituting for V_1 in the above equations:

$$V_0 = R_1 (C_1 \dot{V}_1 + C_2 \dot{V}_2) + V_1$$
$$\Rightarrow V_0 = R_1 \left[C_1 \left(R_2 C_2 \ddot{V}_2 + \dot{V}_2 \right) + C_2 \dot{V}_2 \right] + R_2 C_2 \dot{V}_2 + V_2$$
$$\Rightarrow V_0 = R_1 C_1 R_2 C_2 \ddot{V}_2 + R_1 C_1 \dot{V}_2 + R_1 C_2 \dot{V}_2 + R_2 C_2 \dot{V}_2 + V_2$$

Arrange the equation in the right order, transfer the co-efficient of highest derivative of V_2 onto other terms and add the earlier omitted time dependency entry for the time-dependent:

$$\ddot{V_2}(t) + \frac{R_1 C_1 + R_1 C_2 + R_2 C_2}{R_1 C_1 R_2 C_2} \dot{V_2}(t) + \frac{1}{R_1 C_1 R_2 C_2} V_2(t) = \frac{1}{R_1 C_1 R_2 C_2} V_0(t)$$

For the state space representation, you will notice that there are two energy-storage elements (capacitors) in the circuit. So we are looking for two state variables. This could also be seen from the fact that the above-derived differential equation (input-output) is of second degree.

Select the voltages over the capacitors to be state variables. The following equations can be used to derive the state relations:

$$\dot{V_1} = \frac{1}{C_1} i_3$$

$$\dot{V_2} = \frac{1}{C_2} i_2$$

$$i_1 = \frac{V_0 - V_1}{R_1} = -\frac{1}{R_1} V_1 + \frac{1}{R_1} V_0$$

$$i_2 = \frac{V_1 - V_2}{R_2} = \frac{1}{R_2} V_1 - \frac{1}{R_2} V_2$$

$$i_3 = i_1 - i_2 = -\frac{1}{R_1} V_1 + \frac{1}{R_1} V_0 - \frac{1}{R_2} V_1 + \frac{1}{R_2} V_2 = -\left(\frac{1}{R_1} + \frac{1}{R_2}\right) V_1 + \frac{1}{R_2} V_2 + \frac{1}{R_1} V_0$$

State equations are:

$$\dot{V_1} = -\frac{R_1 + R_2}{R_1 R_2 C_1} V_1 + \frac{1}{R_2 C_1} V_2 + \frac{1}{R_1 C_1} V_0$$

$$\dot{V_2} = \frac{1}{R_2 C_2} V_1 - \frac{1}{R_2 C 2} V_2$$

The matrix format yields a state space representation:

$$\dot{V}(t) = \begin{bmatrix} \dot{V_1}(t) \\ \dot{V_2}(t) \end{bmatrix} = \begin{bmatrix} -\dfrac{R_1 + R_2}{R_1 R_2 C_1} & \dfrac{1}{R_2 C_1} \\ \dfrac{1}{R_2 C_2} & -\dfrac{1}{R_2 C_2} \end{bmatrix} \begin{bmatrix} V_1(t) \\ V_2(t) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{R_1 C_1} \\ 0 \end{bmatrix} V_0(t)$$

$$V_2(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} V(t) = V_2(t)$$

It is seen that it was much easier to build a state space representation than a differential equation that describes the input-output behavior. In later teaching methods, the

transfer function (on the Laplace plane) could be calculated directly from the state space, from which the formation of the input-output differential equation would be quite straightforward.