ELEC-C1230 Control Engineering

Table of Formulas

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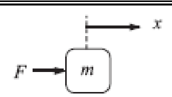
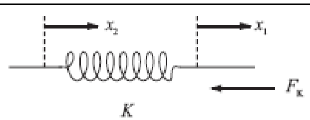
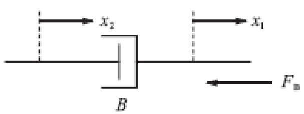
1 Basic components of dynamic models

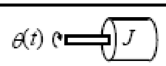
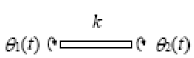
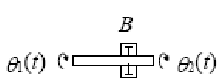
1.1 Electric components

Resistor		$u(t) = Ri(t)$
Coil		$u(t) = L\dfrac{di(t)}{dt}$
Capacitor		$i(t) = C\dfrac{du(t)}{dt}$

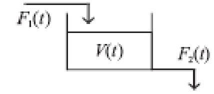
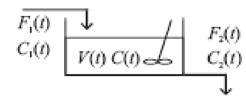
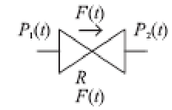
1.2 Linear movement

Mass		$ma(t) = m\dfrac{d^2x(t)}{dt^2}$ a is acceleration
Spring		$F_K = K\Delta x(t)$ $= K(x_1(t) - x_2(t))$
Damper		$F_B = B\Delta v(t) = B\dfrac{d\Delta x(t)}{dt}$ $= B\left(\dfrac{dx_1(t)}{dt} - \dfrac{dx_2(t)}{dt}\right)$ v is speed

1.3 Rotating movement

Moment of inertia		$T_J(t) = J\dfrac{d^2\theta(t)}{dt^2}$
Torque spring		$T_k(t) = k\Delta\theta(t)$ $= k(\theta_1(t) - \theta_2(t))$
Torque damper		$T_b(t) = B\dfrac{d\Delta\theta(t)}{dt}$ $= B\left(\dfrac{d\theta_1(t)}{dt} - \dfrac{d\theta_2(t)}{dt}\right)$

1.4 Flow systems

Vessel		$\dfrac{dV(t)}{dt} = F_1(t) - F_2(t)$
Plug flow		$F_2(t) = F_1(t - T)$ $C_2(t) = C_1(t - T)$
Ideal mixer		$\dfrac{dV(t)}{dt} = F_1(t) - F_2(t)$ $\dfrac{d(C(t)V(t))}{dt} = C_1(t)F_1(t) - C_2(t)F_2(t)$ $C_2(t) = C(t)$
Valve (opening)		$F(t) = A(t)R\sqrt{\Delta P(t)}$ (turbulent) $F(t) = A(t)R\Delta P(t)$ (laminar)

2 State-space representation

System:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_{n-1}y^{(1)}(t) + a_n y(t) = b_1 u^{(n-1)}(t) + \cdots + b_{n-1}u^{(1)}(t) + b_n u(t)$$

General state-space representation:
$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases}$$

Linear state-space representation:
$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

2.1 Linearization

Let the equilibrium state of a general state-space representation be $(\mathbf{x}_s, \mathbf{u}_s)$. The state-space representation linearized in the equilibrium state is

$$\begin{cases} \Delta\mathbf{x}(t) = \dfrac{d\mathbf{f}}{d\mathbf{x}^T}(\mathbf{x}_s, \mathbf{u}_s) \cdot \Delta\mathbf{x}(t) + \dfrac{d\mathbf{f}}{d\mathbf{u}^T}(\mathbf{x}_s, \mathbf{u}_s) \cdot \Delta\mathbf{u}(t) \\ \Delta\mathrm{y}(t) = \dfrac{d\mathbf{g}}{d\mathbf{x}^T}(\mathbf{x}_s, \mathbf{u}_s) \cdot \Delta\mathbf{x}(t) + \dfrac{d\mathbf{g}}{d\mathbf{u}^T}(\mathbf{x}_s, \mathbf{u}_s) \cdot \Delta\mathbf{u}(t) \end{cases}$$

$$\frac{df}{dz^T}(\mathbf{z}) = \begin{bmatrix} \frac{df_1}{dz_1}(\mathbf{z}) & \frac{df_1}{dz_2}(\mathbf{z}) & \cdots & \frac{df_1}{dz_m}(\mathbf{z}) \\ \frac{df_2}{dz_1}(\mathbf{z}) & \frac{df_2}{dz_2}(\mathbf{z}) & & \\ \vdots & & \ddots & \\ \frac{df_n}{dz_1}(\mathbf{z}) & & & \frac{df_n}{dz_m}(\mathbf{z}) \end{bmatrix}$$

2.2 Canonical forms

Controllable canonical form:
$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ 0 & 0 & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} b_n & b_{n-1} & \cdots & b_2 & b_1 \end{bmatrix} x(t) \end{cases}$$

Observable canonical form:
$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -a_1 & 1 & 0 & \cdots & 0 \\ -a_2 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ -a_{n-1} & 0 & & 0 & 1 \\ -a_n & 0 & \cdots & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} x(t) \end{cases}$$

2.3 Solution of the linear state-space representation

$\mathbf{\Phi}(t) = e^{\mathbf{A}t}$, $\qquad \mathbf{\Phi}(t) = e^{\mathbf{A}t} = L^{-1}\left\{(s\mathbf{I} - \mathbf{A})^{-1}\right\}$

$$\begin{cases} \mathbf{x}(t) = \mathbf{\Phi}(t)\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{cases}$$

$$\mathbf{y}(t) = \left(\mathbf{C}\mathbf{\Phi}(t)\mathbf{x}(0)\right) + \left(\mathbf{C}\int_0^t \mathbf{\Phi}(t-\tau)\mathbf{B}\mathbf{u}(\tau)d\tau + \mathbf{D}\mathbf{u}(t)\right) = \mathbf{y}_0(t) + \mathbf{y}_u(t)$$

2.4 Controllability and observability

Controllability matrix:
$$\mathbf{M}_c = \left[\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B} \mid \cdots \mid \mathbf{A}^{n-1}\mathbf{B}\right]$$

Observability matrix:
$$\mathbf{M}_o = \begin{bmatrix} \mathbf{C} \\ \hline \mathbf{CA} \\ \hline \mathbf{CA}^2 \\ \hline \vdots \\ \hline \mathbf{CA}^{n-1} \end{bmatrix}$$

2.5 State control

System:
$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$$

Feedback from the output:
$$\mathbf{u}(t) = \mathbf{T}\mathbf{r}(t) - \mathbf{K}\mathbf{y}(t) = \mathbf{T}\mathbf{r}(t) - \mathbf{KC}\mathbf{x}(t)$$

$$\Rightarrow \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\left(\mathbf{T}\mathbf{r}(t) - \mathbf{KC}\mathbf{x}(t)\right) = \left(\mathbf{A} - \mathbf{BKC}\right)\mathbf{x}(t) + \mathbf{BT}\mathbf{r}(t) = \mathbf{A}^*\mathbf{x}(t) + \mathbf{B}^*\mathbf{r}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$$

Feedback from the state:
$$\mathbf{u}(t) = \mathbf{T}\mathbf{r}(t) - \mathbf{L}\mathbf{x}(t)$$

$$\Rightarrow \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\left(\mathbf{T}\mathbf{r}(t) - \mathbf{L}\mathbf{x}(t)\right) = \left(\mathbf{A} - \mathbf{BL}\right)\mathbf{x}(t) + \mathbf{BT}\mathbf{r}(t) = \mathbf{A}^*\mathbf{x}(t) + \mathbf{B}^*\mathbf{r}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \end{cases}$$

Augmenting the state controller with an integrator:
$$\begin{cases} \mathbf{u}(t) = -\mathbf{L}\mathbf{x}(t) - \mathbf{L}_I\mathbf{x}_I(t) \\ \dot{\mathbf{x}}_I(t) = -\mathbf{C}\mathbf{x}(t) + \mathbf{y}_{ref}(t) \end{cases}$$

$$\Rightarrow \begin{cases} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \hline \dot{\mathbf{x}}_I(t) \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A} - \mathbf{BL} & -\mathbf{BL}_I \\ \hline -\mathbf{C} & \mathbf{0} \end{array}\right] \begin{bmatrix} \mathbf{x}(t) \\ \hline \mathbf{x}_I(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \hline \mathbf{I} \end{bmatrix} \mathbf{y}_{ref}(t) \\ \mathbf{y}(t) = \left[\mathbf{C} \mid \mathbf{0}\right] \begin{bmatrix} \mathbf{x}(t) \\ \hline \mathbf{x}_I(t) \end{bmatrix} \end{cases} \Rightarrow \begin{cases} \dot{\mathbf{x}}^*(t) = \mathbf{A}^*\mathbf{x}^*(t) + \mathbf{B}^*\mathbf{y}_{ref}(t) \\ \mathbf{y}(t) = \mathbf{C}^*\mathbf{x}^*(t) \end{cases}$$

Characteristic equation:
$$\det\left(s\mathbf{I} - \mathbf{A}^*\right) = \det\left[\begin{array}{c|c} s\mathbf{I} - \mathbf{A} + \mathbf{BL} & \mathbf{BL}_I \\ \hline \mathbf{C} & s\mathbf{I} \end{array}\right] = 0$$

2.6 State estimation

$$\begin{cases} \mathbf{x}(t) & \text{actual state} \\ \hat{\mathbf{x}}(t) & \text{state estimate} \\ \tilde{\mathbf{x}}(t) & \text{error of the estimate} \end{cases}$$

State observer:

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{K}\hat{\mathbf{e}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{K}\left(\mathbf{y}(t) - \mathbf{C}\hat{\mathbf{x}}(t)\right)$$
$$= \left(\mathbf{A} - \mathbf{K}\mathbf{C}\right)\hat{\mathbf{x}}(t) + \mathbf{B}\mathbf{u}(t) + \mathbf{K}\mathbf{y}(t)$$

Dynamics of the estimation error:

$$\dot{\tilde{\mathbf{x}}}(t) = \left(\mathbf{A} - \mathbf{K}\mathbf{C}\right)\tilde{\mathbf{x}}(t) = \mathbf{A}^{*}\tilde{\mathbf{x}}(t)$$

2.7 *State estimation and state control combined*

$$\begin{cases} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \hline \dot{\tilde{\mathbf{x}}}(t) \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{A} - \mathbf{B}\mathbf{L} & \mathbf{B}\mathbf{L} \\ \hline 0 & \mathbf{A} - \mathbf{K}\mathbf{C} \end{array}\right] \begin{bmatrix} \mathbf{x}(t) \\ \hline \tilde{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{B}\mathbf{T} \\ \hline 0 \end{bmatrix} \mathbf{r}(t) \\ \mathbf{y}(t) = [\mathbf{C} \mid 0] \begin{bmatrix} \mathbf{x}(t) \\ \hline \tilde{\mathbf{x}}(t) \end{bmatrix} \end{cases}$$

Characteristic equation:

$$\det(s\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{L}) \cdot \det(s\mathbf{I} - \mathbf{A} + \mathbf{K}\mathbf{C}) = 0$$

3 Transfer function

Transfer function: $G(s) = \dfrac{Y(s)}{U(s)}$

Output: $y(t) = L^{-1}\{G(s) \cdot L\{u(t)\}\}$

Static gain: $\overline{k} = \lim_{s \to 0}\{G(s)\}$

Transfer function from a state-space representation: $\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$

3.1 *First order dynamics*

$$\tau\dot{y}(t) + y(t) = Ku(t) \qquad G(s) = \frac{K}{\tau s + 1}$$

Impulse response: $y(t) = \dfrac{K}{\tau} e^{-\frac{t}{\tau}}$

Step response: $y(t) = K\left(1 - e^{-\frac{t}{\tau}}\right)$

3.2 *Second order dynamics*

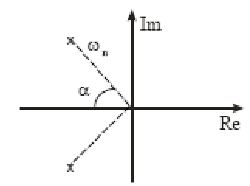
$$\ddot{y}(t) + 2\zeta\omega_n\dot{y}(t) + \omega_n^2 = K\omega_n^2 u(t) \qquad G(s) = \frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Natural frequency:

ω_n

Damping coefficient:

$\zeta = \cos(\alpha)$



Time domain:

Impulse response: $y(t) = \dfrac{K\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left(\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\right)$

Step response: $y(t) = K\left(1 - e^{-\zeta\omega_n t}\left(\cos\left(\omega_n\sqrt{1-\zeta^2}\,t\right) + \dfrac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right)\right)\right)$

$$= K\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left(\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \arccos(\zeta)\right)\right)\right)$$

Overshoot of the step response:

$$\%\mathrm{OS} = 100 e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

Time of peak:

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Settling time of the step response:

$$t_s(2\%) \approx \begin{cases} \dfrac{4}{\zeta\omega_n}, & \zeta \le 0.88 \\ \dfrac{10\zeta - 4.2}{\omega_n}, & 0.88 < \zeta \le 1.4 \end{cases} \qquad t_s(5\%) \approx \begin{cases} \dfrac{3}{\zeta\omega_n}, & \zeta \le 0.83 \\ \dfrac{7\zeta - 2.2}{\omega_n}, & 0.83 < \zeta \le 1.4 \end{cases}$$

Frequency domain:

Frequency of the resonance peak:

$$\omega_r = \omega_n\sqrt{1 - 2\zeta^2}$$

Magnitude of the resonance peak:

$$M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

3 dB bandwidth:

$$\mathrm{BW} = \omega_n\left[(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}\right]^{\frac{1}{2}}$$

Static error:

$$G_{OL}(s)=\frac{P(s)}{Q(s)}=\frac{P(s)}{s^k\cdot Q^*(s)}$$

$Q^*(s)=0$ has no roots in the origin

$k=0$: $K_p=\lim_{s\to 0}\{G_{OL}(s)\}$

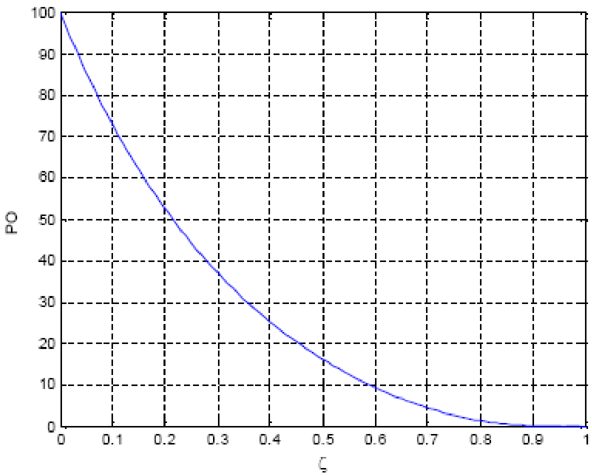
$k=1$: $K_v=\lim_{s\to 0}\{s\cdot G_{OL}(s)\}$

$k=2$: $K_a=\lim_{s\to 0}\{s^2\cdot G_{OL}(s)\}$

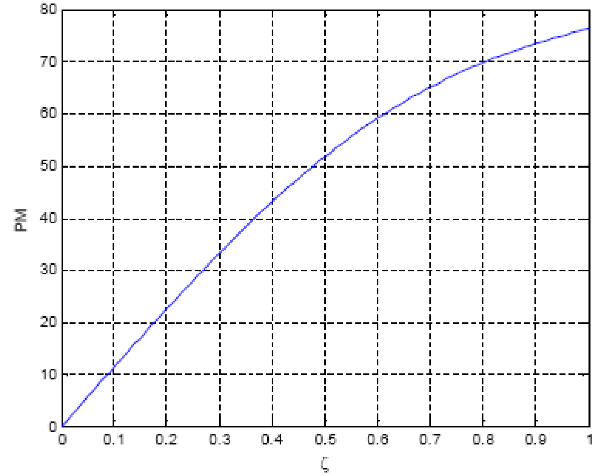
	Input		
k	a_0	$a_1 t$	$a_2 t^2$
0	$\frac{a_0}{1+K_p}$	$\pm\infty$	$\pm\infty$
1	0	$\frac{a_1}{K_v}$	$\pm\infty$
2	0	0	$\frac{2a_2}{K_a}$

Static error

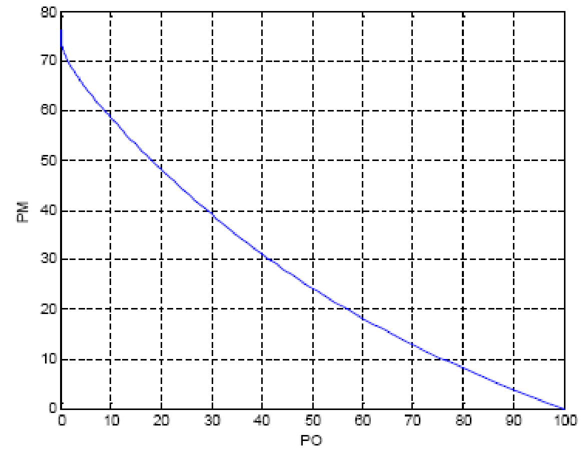
%-overshoot of the unit step response and phase margin as function of the damping coefficient:



PO: Overshoot in %



PM: Phase margin (degrees)



PO: Overshoot in %

PM: Phase margin (degrees)

4 Poles and zeros

Transfer function

$$G(s)=\frac{b_1 s^{n_z-1}+b_2 s^{n_z-2}+\cdots+b_{n_z-2}s^2+b_{n_z-1}s+b_{n_z}}{s^{n_p}+a_1 s^{n_p-1}+a_2 s^{n_p-2}+\cdots+a_{n_p-2}s^2+a_{n_p-1}s+a_{n_p}}=\frac{P(s)}{Q(s)}$$

State-space representation

$$\begin{cases} Q(s)=\det(s\mathbf{I}-\mathbf{A}) \\ P(s)=\mathbf{C}\,\mathrm{adj}(s\mathbf{I}-\mathbf{A})\mathbf{B}+\mathbf{D}\det(s\mathbf{I}-\mathbf{A}) \end{cases}$$

Characteristic polynomial: $Q(s)=0$

4.1 Root locus

Characteristic polynomial: $1+KG_{OL}(s)=0$

Root locus: $G_{OL}(s_{Pi})=-1$

$$|G_{OL}(s_{Pi})|=1,\quad \sphericalangle\{G_{OL}(s_{Pi})\}=\pi+k\cdot 2\pi,\quad k=0,\pm1,\pm2,\ldots$$

Angles of the asymptotes:

$$\alpha=\pm\frac{(2k+1)\cdot 180}{P-Z}$$

Intersection points of the asymptotes on the real axis:

$$\delta=\frac{\sum_{i=1}^{P}s_{pi}-\sum_{i=1}^{Z}s_{zi}}{P-Z}$$

Joining and division points of the branches of the root locus:

$$\frac{dK}{ds}=0$$

4.2 Routh array

$$a_0 s^n+a_1 s^{n-1}+\cdots+a_{n-1}s+a_n$$

s^n	a_0	a_2	a_4	a_6	a_8	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	a_9	$\cdots$
s^{n-2}	b_0	b_2	b_4	b_6	$\cdots$	
s^{n-3}	b_1	b_3	b_5	b_7	$\cdots$	
s^{n-4}	c_0	c_2	c_4	$\cdots$		
s^{n-5}	c_1	c_3	c_5	$\cdots$		
$\vdots$	$\vdots$					
s^1	z_0					
s^0	z_1					

$$b_0=\frac{-1}{a_1}\begin{vmatrix}a_0 & a_2\\ a_1 & a_3\end{vmatrix},\quad b_2=\frac{-1}{a_1}\begin{vmatrix}a_0 & a_4\\ a_1 & a_5\end{vmatrix},\quad b_4=\frac{-1}{a_1}\begin{vmatrix}a_0 & a_6\\ a_1 & a_7\end{vmatrix},\quad\cdots$$

$$b_1=\frac{-1}{b_0}\begin{vmatrix}a_1 & a_3\\ b_0 & b_2\end{vmatrix},\quad b_3=\frac{-1}{b_0}\begin{vmatrix}a_1 & a_5\\ b_0 & b_4\end{vmatrix},\quad b_5=\frac{-1}{b_0}\begin{vmatrix}a_1 & a_7\\ b_0 & b_6\end{vmatrix},\quad\cdots$$

$$b_1=\frac{-1}{b_1}\begin{vmatrix}b_0 & b_2\\ b_1 & b_3\end{vmatrix},\quad b_3=\frac{-1}{b_1}\begin{vmatrix}b_0 & b_4\\ b_1 & b_5\end{vmatrix},\quad b_5=\frac{-1}{b_1}\begin{vmatrix}b_0 & b_6\\ b_1 & b_7\end{vmatrix},\quad\cdots$$

$\vdots$

$z_1=a_n$

5 Frequency response

Frequency function:

$$F(\omega)=G(j\omega)=\mathrm{Re}\{G(j\omega)\}+j\cdot\mathrm{Im}\{G(j\omega)\}\quad\Rightarrow\quad z=R+j\cdot X$$

$$\begin{cases} |z|=\sqrt{R^2+X^2} \\ \angle\{z\}=\arctan(X/R)+n\pi \qquad (n=0 \text{ jos } R>0,\ n=1 \text{ jos } R<0) \end{cases}$$

Amplitude ratio A and phase difference φ:

$$A=\frac{A_y}{A_u}=|G(j\omega)|=\sqrt{\mathrm{Re}\{G(j\omega)\}^2+\mathrm{Im}\{G(j\omega)\}^2}$$

$$\varphi=\angle\{G(j\omega)\}=\arctan\left(\mathrm{Im}\{G(j\omega)\}/\mathrm{Re}\{G(j\omega)\}\right)+n\pi$$

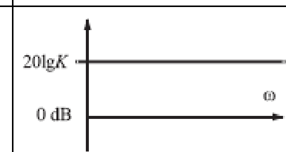
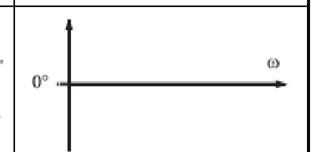
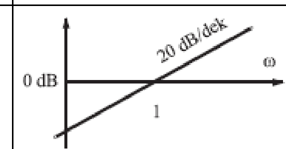
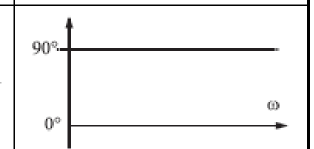
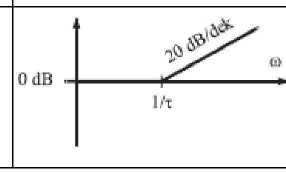
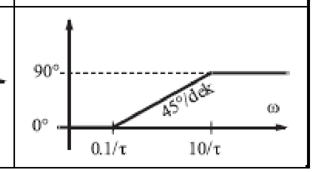
$$(n=0 \text{ jos } \mathrm{Re}\{G(j\omega)\}>0,\ n=1 \text{ jos } \mathrm{Re}\{G(j\omega)\}<0)$$

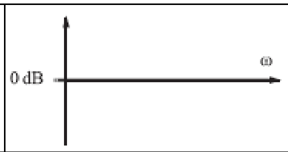
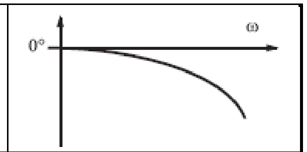
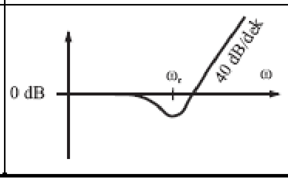
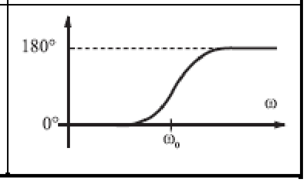
Euler's formula:

$$e^{Aj}=\cos(A)+\sin(A)j$$

$$e^{-Bj}=\cos(B)-\sin(B)j$$

5.1 Basic components of the Bode diagram

$G(j\omega)$	$20\cdot\lg\lvert G(j\omega)\rvert dB$	$\angle\{G(j\omega)\}$
K $(K\geq 0)$		
s		
$\tau s+1$ $(\tau>0)$		

$e^{-\tau s}$ $(\tau > 0)$		
$(s^2 + 2\zeta\omega_n s + \omega_n^2)/\omega_n^2$ $(0 \le \xi < 1)$		

$\omega_r = \omega_n\sqrt{1-2\zeta^2}$

If the component to be drawn is in the denominator, the gain and phase curves must be mirrored with respect to zero.

6 Controllers

6.1 PID controller

$$\begin{cases} u(t) = K_P\left(e(t) + \dfrac{1}{T_I}\displaystyle\int_0^t e(\tau)d\tau + T_D\dfrac{de(t)}{dt}\right) \\ u(t) = K_P e(t) + K_I\displaystyle\int_0^t e(\tau)d\tau + K_D\dfrac{de(t)}{dt} \end{cases}$$

$$\begin{cases} K_I = \dfrac{K_P}{T_I} \\ K_D = K_P T_D \end{cases}$$

6.2 Phase lead compensator

$$G_{lead}(s) = K\frac{\frac{1}{\omega_0}s+1}{\frac{1}{k\omega_0}s+1}$$

$$\omega_m = \sqrt{k}\,\omega_0$$

$$\sin(\phi_m) = \frac{k-1}{k+1}$$

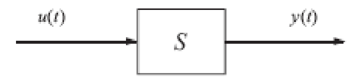
6.3 Phase lag compensator

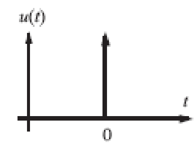
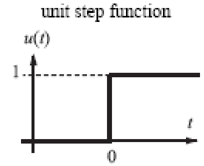
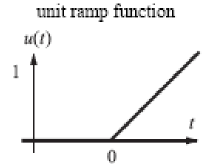
$$G_{lag}(s) = K\frac{\frac{1}{k\omega_0}s+1}{\frac{1}{\omega_0}s+1}$$

6.4 Phase lead-lag compensator

$$G_{lead-lag}(s) = K\frac{\frac{1}{\omega_1}s+1}{\frac{1}{k_1\omega_1}s+1}\cdot\frac{\frac{1}{k_2\omega_2}s+1}{\frac{1}{\omega_2}s+1}$$

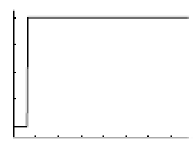
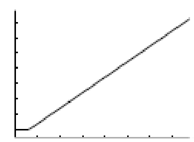
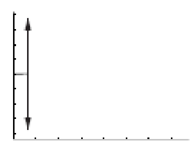
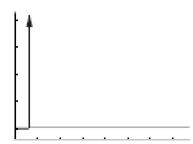
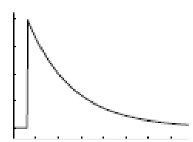
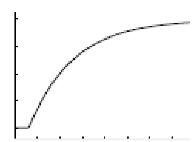
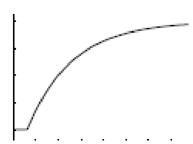
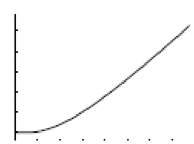
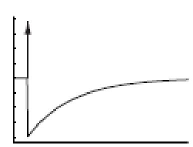
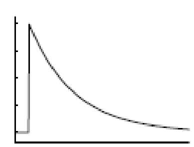
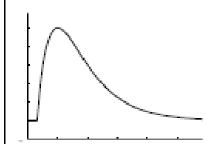
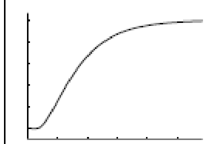
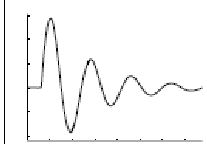
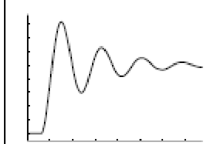
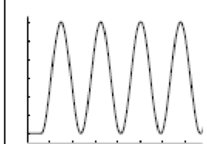
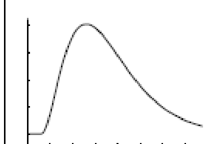
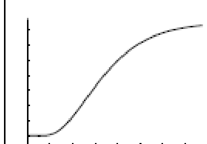
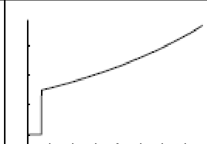
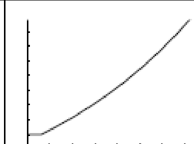
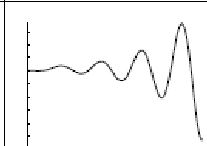
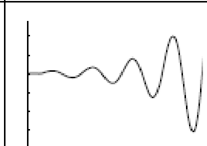
7 Test functions



Test function	Time domain	Laplace transform
unit impulse function 	$u(t) = \delta(t) = \begin{cases} \infty, & t = 0_+ \\ 0, & \text{otherwise} \end{cases}$ $\int_{-\infty}^{\infty} u(t)\,dt = 1$	$U(s) = 1$
unit step function 	$u(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$	$U(s) = \frac{1}{s}$
unit ramp function 	$u(t) = \begin{cases} 0, & t \le 0 \\ t, & t > 0 \end{cases}$	$U(s) = \frac{1}{s^2}$

8 Impulse and step responses of simple systems

$\tau > 0,\ K > 0,\ 0 < \zeta < 1,\ \omega_n \neq 0$

	Transfer function	Impulse response	Step response
1.	$\frac{K}{s}$		
2.	Ks		
3.	$\frac{K}{\tau s+1}$		
4.	$\frac{K}{s(\tau s+1)}$		
5.	$\frac{Ks}{\tau s+1}$		
6.	$\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}$		
7.	$\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$		
8.	$\frac{K\omega_n^2}{s^2+\omega_n^2}$		
9.	$\frac{K}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}$		
10.	$\frac{K}{\tau s-1}$		
11.	$\frac{K\omega_n^2}{s^2-2\zeta\omega_n s+\omega_n^2}$		

9 Partial fraction expansion

$$Y(s)=\frac{P(s)}{(s-r_1)(s-r_2)\ldots(s-r_n)}=\frac{K_1}{(s-r_1)}+\frac{K_2}{(s-r_2)}+\ldots+\frac{K_n}{(s-r_n)},\ r_i\neq r_j\ i\neq j$$

$$Y(s)=\frac{P(s)}{(s-r)^q(s-r_1)\ldots(s-r_n)}$$

$$=\frac{C_q}{(s-r)^q}+\frac{C_{q-1}}{(s-r)^{q-1}}+\ldots+\frac{C_1}{(s-r)}+\frac{K_1}{(s-r_1)}+\ldots+\frac{K_n}{(s-r_n)}$$

Heaviside method

$$K_i=\lim_{s\to r_i}\{(s-r_i)\cdot Y(s)\}\qquad i=1,2,\ldots,n$$

$$C_q=\lim_{s\to r}\left\{(s-r)^q\cdot Y(s)\right\}$$

$$C_{q-1}=\lim_{s\to r}\left\{\frac{d}{ds}\left[(s-r)^q\cdot Y(s)\right]\right\}$$

$$\vdots$$

$$C_{q-k}=\lim_{s\to r}\left\{\frac{1}{k!}\frac{d^k}{ds^k}\left[(s-r)^q\cdot Y(s)\right]\right\}$$

10 Laplace transform

Definition: $F(s)=L\{f(t)\}=\int_0^\infty f(t)e^{-st}dt$

10.1 Laplace transform theorems

Laplace transform	Time domain function
$F(s)$	$f(t)$
$C_1F_2(s)+C_2F_2(s)$	$C_1f_2(t)+C_2f_2(t)$
$F(s+a)$	$e^{-at}f(t)$
$e^{-as}F(s)$	$\begin{cases}0, & t\le a\\ f(t-a), & t>a\end{cases}$
$\frac{1}{a}F\left(\frac{s}{a}\right)$	$f(at)$
$F_1(s)F_2(s)$	$\int_0^t f_1(\tau)f_2(t-\tau)d\tau$
$sF(s)-f(0)$	$f'(t)$
$s^nF(s)-\left[s^{n-1}f(0)+\ldots+f^{(n-1)}(0)\right]$	$f^{(n)}(t)$

10.2 Laplace transforms and time domain functions

Laplace transform	Time domain function
1	$\delta(t)$
$1/s$	1
$1/s^2$	t
$1/s^{n+1}$	$t^n/n!$
$\frac{1}{s+a}$	e^{-at}
$\frac{1}{(s+a)^{n+1}}$	$\frac{t^n e^{-at}}{n!}$
$\frac{1}{s(s+a)}$	$\frac{1}{a}\left(1-e^{-at}\right)$
$\frac{1}{(s+a)(s+b)}$	$\frac{1}{a-b}\left(e^{-bt}-e^{-at}\right)$
$\frac{1}{s(s+a)(s+b)}$	$\frac{1}{ab}+\frac{1}{ab(b-a)}\left(ae^{-bt}-be^{-at}\right)$
$\frac{a}{s^2+a^2}$	$\sin(at)$
$\frac{s}{s^2+a^2}$	$\cos(at)$
$\frac{a}{(s+b)^2+a^2}$	$e^{-bt}\sin(at)$
$\frac{s+b}{(s+b)^2+a^2}$	$e^{-bt}\cos(at)$
$\frac{s+a}{s+b}$	$\delta(t)+(a-b)e^{-bt}$

If limits for $f(t)$ and $F(s)$ exist, they satisfy:

$$\lim_{s\to 0}\{sF(s)\}=\lim_{t\to\infty}\{f(t)\}\qquad \lim_{s\to\infty}\{sF(s)\}=\lim_{t\to 0}\{f(t)\}$$