1. **a.** Taking the Laplace transform of the differential equation assuming zero initial values. (This corresponds to a situation where the system starts at a constant operation point (y_s, u_s) after which the input variable changes. In a linear system it is the same whether the equilibrium point is initially (0,0) or some other value pair. The dynamics of the response are similar):

$$\ddot{y}(t)+3\dot{y}(t)+2y(t)=3u(t)$$
$$\Rightarrow s^2Y(s)+3sY(s)+2Y(s)=\left(s^2+3s+2\right)Y(s)=3U(s)$$
$$\Rightarrow G(s)=\frac{Y(s)}{U(s)}=\frac{3}{s^2+3s+2}$$

b. $u(t)$ is unit impulse function $\delta(t)$.

$$U(s)=L\{\delta(t)\}=1$$

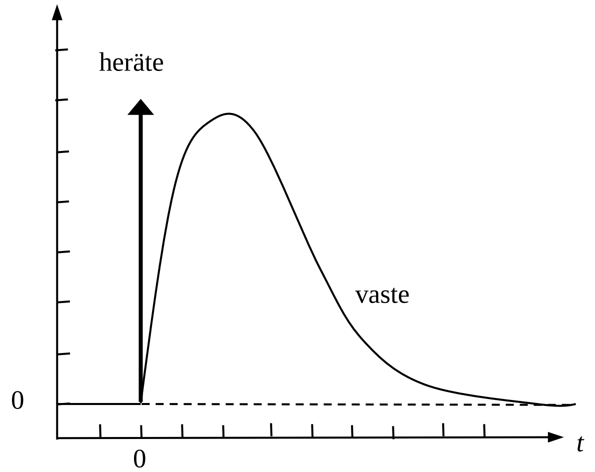
Deriving the transfer function from the impulse response of $Y(s)$:

$$Y(s)=G(s)U(s)=\frac{3}{s^2+3s+2}\cdot 1=\frac{3}{s^2+3s+2}$$

Taking Laplace inverse of $Y(s)$:

$$y(t)=L^{-1}\{Y(s)\}=L^{-1}\left\{\frac{3}{s^2+3s+2}\right\}=3\cdot L^{-1}\left\{\frac{1}{(s+1)(s+2)}\right\}$$
$$=\frac{3}{(2-1)}\left(e^{-t}-e^{-2t}\right)=3\left(e^{-t}-e^{-2t}\right)$$

Plotting the output of an impulse response on a time graph:

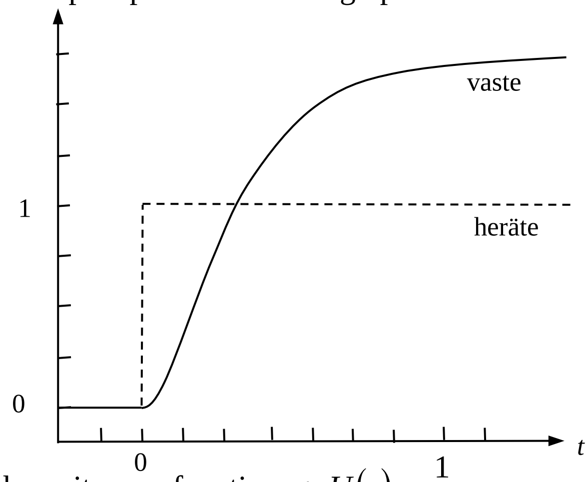


c. $u(t)$ is a unit step function $\Rightarrow U(s) = \frac{1}{s}$

$$Y(s) = G(s)U(s) = \frac{3}{s^2+3s+2}\cdot\frac{1}{s} = \frac{3}{s(s^2+3s+2)} = \frac{3}{s(s+1)(s+2)}$$

$$y(t) = \frac{3}{2} - 3e^{-t} + \frac{3}{2}e^{-2t}$$

Plotting the step response on a time graph:



d. $u(t)$ is the unit ramp function $\Rightarrow U(s) = \frac{1}{s^2}$

$$Y(s) = G(s)U(s) = \frac{3}{s^2+3s+2}\cdot\frac{1}{s^2} = \frac{3}{s^2(s^2+3s+2)} = \frac{3}{s^2(s+1)(s+2)}$$

There is no direct inverse transformation of Y(s) from the table. It has to be broken down into partial fractions as follows:

$$Y(s) = \frac{3}{s^2(s+1)(s+2)} = \frac{A_1}{s^2} + \frac{A_2}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\begin{cases} A_1 = \lim\limits_{s\to 0}\left\{\dfrac{3}{s^2(s+1)(s+2)}\cdot s^2\right\} = \lim\limits_{s\to 0}\left\{\dfrac{3}{(s+1)(s+2)}\right\} = \dfrac{3}{1\cdot 2} = \dfrac{3}{2} \\ A_2 = \lim\limits_{s\to 0}\left\{\dfrac{d}{ds}\left[\dfrac{3}{s^2(s+1)(s+2)}\cdot s^2\right]\right\} = \lim\limits_{s\to 0}\left\{\dfrac{d}{ds}\left[\dfrac{3}{(s+1)(s+2)}\right]\right\} \\ \qquad = \lim\limits_{s\to 0}\left\{-\dfrac{3(2s+3)}{(s^2+3s+2)^2}\right\} = -\dfrac{3^2}{2^2} = -\dfrac{9}{4} \\ B = \lim\limits_{s\to -1}\left\{\dfrac{3}{s^2(s+1)(s+2)}\cdot(s+1)\right\} = \lim\limits_{s\to -1}\left\{\dfrac{3}{s^2(s+2)}\right\} = 3 \\ C = \lim\limits_{s\to -2}\left\{\dfrac{3}{s^2(s+1)(s+2)}\cdot(s+2)\right\} = \lim\limits_{s\to -2}\left\{\dfrac{3}{s^2(s+1)}\right\} = -\dfrac{3}{4} \end{cases}$$

$$\Rightarrow Y(s) = \frac{3}{2}\frac{1}{s^2} - \frac{9}{4}\frac{1}{s} + 3\frac{1}{s+1} - \frac{3}{4}\frac{1}{s+2}$$

Inverse transformation according to general rules:

$$\Rightarrow y(t) = \frac{3}{2}t - \frac{9}{4} + 3e^{-t} - \frac{3}{4}e^{-2t}$$

2. **a.**

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 4 \\ 10 \end{bmatrix}u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x}(t) \end{cases}$$

The differential presentation only includes an input $u(t)$ and response $y(t)$. So, we want to get rid of the state variables $x_1(t)$, and $x_2(t)$ for the differential representation. Let's first write the matrices in expanded form:

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 4 \\ 10 \end{bmatrix}u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$

By using the matrix algebra rules, we get the equations:

$$\begin{cases} \dot{x}_1(t) = -5x_1(t) + x_2(t) + 4u(t) \\ \dot{x}_2(t) = -6x_1(t) + 10u(t) \\ y(t) = x_1(t) \end{cases}$$

Substitute the last equation in 1st and 2nd:

$$\begin{cases} \dot{y}(t) = -5y(t) + x_2(t) + 4u(t) \\ \dot{x}_2(t) = -6y(t) + 10u(t) \end{cases}$$

Expressing $x_2(t)$ in terms of other variables:

$$x_2(t) = \dot{y}(t) + 5y(t) - 4u(t)$$
$$\Rightarrow \dot{x}_2(t) = \ddot{y}(t) + 5\dot{y}(t) - 4\dot{u}(t)$$

Substituting these in the other equations, we get the differential form:

$$\ddot{y}(t) + 5\dot{y}(t) - 4\dot{u}(t) = -6y(t) + 10u(t)$$

which can be rewritten as:

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 4\dot{u}(t) + 10u(t)$$

Taking the Laplace transform to get the transfer function:

$$s^2Y(s) + 5sY(s) + 6Y(s) = 4sU(s) + 10U(s)$$
$$G(s) = \frac{Y(s)}{U(s)} = \frac{4s+10}{s^2+5s+6}$$

b.

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \mathbf{x}(t) \end{cases}$$

For the benefit of the reader, a second method is shown where the transfer function is obtained directly from state space representation:

$$G(s) = \mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}$$

with

$$G(s) = \begin{bmatrix} 2 & 1 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 1 \end{bmatrix} \left(\begin{bmatrix} s+2 & 0 \\ 0 & s+3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

2×2-inverse matrix can be determined using the formula:

$$\mathbf{A}^{-1} = \det{}^{-1}\mathbf{A} \cdot \mathrm{adj}\mathbf{A} = \det{}^{-1}\mathbf{A} \cdot \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\frac{1}{(s+2)(s+3)} \cdot \begin{bmatrix} s+3 & 0 \\ 0 & s+2 \end{bmatrix}$$

Substituting the value of $(s\mathbf{I}\text{-}\mathbf{A})^{-1}$-matrix and performing the corresponding calculations:

$$G(s) = \frac{\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} s+3 & 0 \\ 0 & s+2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{(s+2)(s+3)} = \frac{\begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} s+3 \\ 2(s+2) \end{bmatrix}}{(s+2)(s+3)}$$

$$= \frac{2s+6+2s+4}{(s+2)(s+3)} = \frac{4s+10}{(s+2)(s+3)}$$

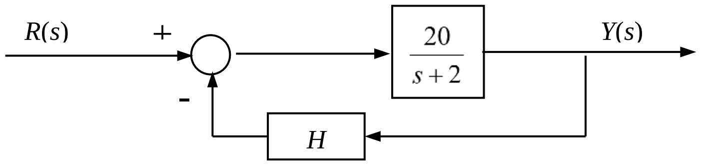
The transfer function now provides an easy method to obtain the differential equation using Inverse Laplace transformation:

$$G(s) = \frac{4s+10}{(s+2)(s+3)} = \frac{Y(s)}{U(s)}$$

$$s^2Y(s) + 5sY(s) + 6Y(s) = 4sU(s) + 10U(s)$$

$$\ddot{y}(t) + 5\dot{y}(t) + 6y(t) = 4\dot{u}(t) + 10u(t)$$

3. Calculate the system transfer function of the system below ($Y(s)/R(s)$):



(Note that even if H here is a scalar, it could be an arbitrary transfer function $H(s)$.)

Assigning the plant transfer function $G(s)$ as:

$$G(s) = \frac{20}{s+2}$$

The expressions for error function $E(s)$ and output function $Y(s)$ can be derived from the system shown as :

$$E(s) = R(s) - H(s)Y(s)$$

$$Y(s) = G(s)E(s)$$

Substituting the first equation in the second one, we get:

$$
\begin{aligned}
&Y(s) = G(s)\big(R(s) - H(s)Y(s)\big) = G(s)R(s) - G(s)H(s)Y(s)\\
&\Rightarrow Y(s) + G(s)H(s)Y(s) = G(s)R(s)\\
&\Leftrightarrow \big(1 + G(s)H(s)\big)Y(s) = G(s)R(s)\\
&\Rightarrow \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)} = G_{tot}(s)
\end{aligned}
$$

This is the well-known form of the negative feedback control loop transfer function. The result is worth understanding and knowing for proper understanding of control theory

If $H = 0.4$, then $G_{tot}(s) = \frac{20}{s+10} \Rightarrow$ and pole $s = -10$.

If $H = 0.9$, then $G_{tot}(s) = \frac{20}{s+20} \Rightarrow$ and pole $s = -20$.

Calculate Impulse Responses:

Impulse (i.e. Dirac delta function $\delta(t)$) Laplace transformation: $R(s) = 1$

$\Rightarrow \quad Y(s) = G_{tot}(s)R(s) = \frac{20}{s+10} \Rightarrow y(t) = L^{-1}\{Y(s)\} = 20\frac{1}{s+10} = 20e^{-10t}$

$Y(s) = G_{tot}(s)R(s) = \frac{20}{s+20} \Rightarrow y(t) = L^{-1}\{Y(s)\} = 20\frac{1}{s+20} = 20e^{-20t}$

Calculate Step Responses:

Unit Step (i.e. output is one when time > 0) Laplace transformation $R(s) = \frac{1}{s}$

$\Rightarrow Y(s) = G_{tot}(s)R(s) = \frac{1}{s}\cdot\frac{20}{s+10} \Rightarrow y(t) = L^{-1}\{Y(s)\} = 20\frac{1}{s(s+10)} = \frac{20}{10}\left(1-e^{-10t}\right) = 2-2e^{-10t}$

$Y(s) = G_{tot}(s)R(s) = \frac{1}{s}\cdot\frac{20}{s+20} \Rightarrow y(t) = L^{-1}\{Y(s)\} = 20\frac{1}{s(s+20)} = \frac{20}{20}\left(1-e^{-20t}\right) = 1-e^{-20t}$

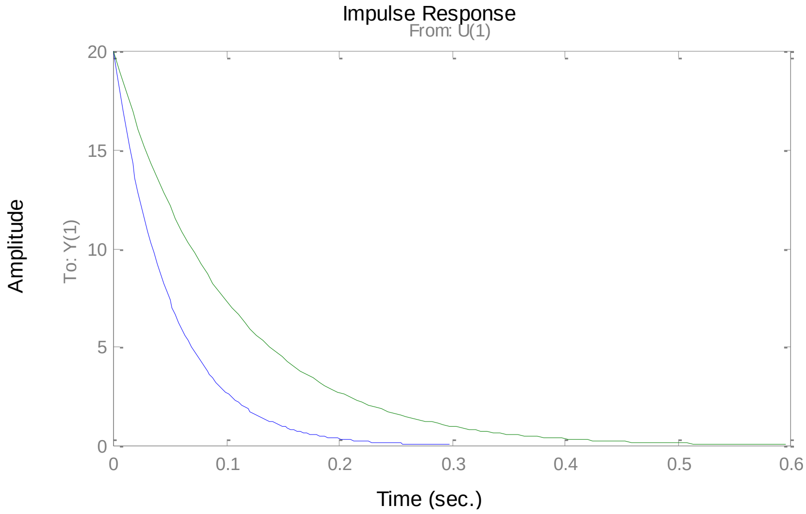
Let's do the same with Matlab:

The polynomial coefficients are given to Matlab in vector form.

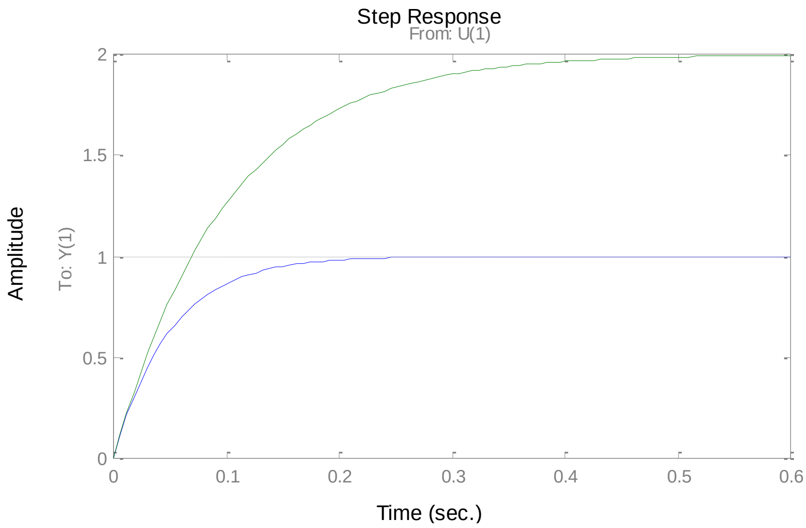
```
>> G=tf(20,[1 2])
Transfer function:
 20
-----
s + 2
>> H=0.9; Gtot=feedback(G,H)
Transfer function:
  20
------
s + 20
```

Simulated responses:

```
>> impulse(Gtot);hold on;impulse(Gtot2)
```




```
>> step(Gtot,0.6);hold on;step(Gtot2,0.6)
```



4. Forming concentration balances for ideal mixers according to given hint:

$$\begin{cases} \dfrac{d\left(V_1 C_1(t)\right)}{dt} = QC_i(t) - QC_1(t) \\ \dfrac{d\left(V_2 C_2(t)\right)}{dt} = QC_1(t) - QC_2(t) \end{cases}$$

$$\begin{cases} V_1 \dfrac{dC_1(t)}{dt} = V_1 \dot{C}_1(t) = Q(C_i(t) - C_1(t)) \\ V_2 \dfrac{dC_2(t)}{dt} = V_2 \dot{C}_2(t) = Q(C_1(t) - C_2(t)) \end{cases}$$

$$\begin{cases} \dot{C}_1(t) = \dfrac{Q}{V_1}(C_i(t) - C_1(t)) \\ \dot{C}_2(t) = \dfrac{Q}{V_2}(C_1(t) - C_2(t)) \end{cases}$$

Substituting numerical values:

$$\begin{cases} \dot{C}_1(t) = 2(C_i(t) - C_1(t)) = 2C_i(t) - 2C_1(t) \\ \dot{C}_2(t) = 5(C_1(t) - C_2(t)) = 5C_1(t) - 5C_2(t) \end{cases}$$

The above-derived equations are easy to present as a state space representation with a physical significance for the state variables. Choosing an input as $u(t) = C_i(t)$, output as $y(t) = C_2(t)$ and states as $x_1(t) = C_1(t)$ and $x_2(t) = C_2(t)$ we obtain the state space representation as follows:

$$\begin{cases} \dot{x}_1(t) = 2u(t) - 2x_1(t) \\ \dot{x}_2(t) = 5x_1(t) - 5x_2(t) \\ y(t) = x_2(t) \end{cases}$$

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 5 & -5 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{cases}$$

The transfer function can be resolved using a variety of methods, such as a newly-described conversion from state space directly:

$$G(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Select the transfer function in such a way that the Laplace transformed equations represent the required input and output quatities while eliminating the intermediate state variables.

- $U(s) = C_i(s)$ ja $Y(s) = C_2(s)$. Eliminating $C_1(s)$.
- $G(s) = \dfrac{Y(s)}{U(s)} = \dfrac{C_2(s)}{C_i(s)}$

$$\begin{cases} sC_1(s) = 2C_i(s) - 2C_1(s) \\ sC_2(s) = 5C_1(s) - 5C_2(s) \end{cases}$$

$$\begin{cases} C_1(s) = \dfrac{2}{s+2} C_i(s) \\ sC_2(s) = 5C_1(s) - 5C_2(s) \end{cases}$$

Substitute first equation in second:

$$sC_2(s) = 5\frac{2}{s+2}C_i(s) - 5C_2(s)$$
$$(s+5)C_2(s) = \frac{10}{s+2}C_i(s)$$
$$G(s) = \frac{C_2(s)}{C_i(s)} = \frac{10}{(s+2)(s+5)}$$

a. Unit Impulse response: $C_i(s) = 1$

$$C_2(t) = L^{-1}\{C_2(s)\} = L^{-1}\{G(s)C_i(s)\} = L^{-1}\left\{\frac{10}{(s+2)(s+5)}\cdot 1\right\} = \frac{10}{3}\left(e^{-2t} - e^{-5t}\right)$$

b. Unit Step response: $C_2(s) = \frac{1}{s}$

$$C_2(t) = L^{-1}\left\{\frac{10}{(s+2)(s+5)}\cdot\frac{1}{s}\right\} = 1 - \frac{5}{3}e^{-2t} + \frac{2}{3}e^{-5t}$$

c. Steady state value of step response:

$$\lim_{t\to\infty}\{C_2(t)\} = \lim_{t\to\infty}\left\{1 - \frac{5}{3}e^{-2t} + \frac{2}{3}e^{-5t}\right\} = 1$$

Same result is also obtained from final value theorm (FVT):

$$\lim_{t\to\infty}\{C_2(t)\} = \lim_{s\to 0}\{C_2(s)\cdot s\} = \lim_{s\to 0}\left\{\frac{10}{(s+2)(s+5)}\cdot\frac{1}{s}\cdot s\right\} = \frac{10}{10} = 1$$

d. From (c), we see that the values of step input and response are the same at steady state, which means that the static gain of the system is one. In this example, it means that whenever we change the concentration ratio of the input flow, the concentration of the output flow comes to that value after a long time.

Can also be confirmed by using direct static gain formula:

$$\bar{k} = \lim_{s\to 0}\{G(s)\} = 1$$

e. The weight function is the same as time domain impulse response. It is given by:

$$g(t) = \frac{10}{3}\left(e^{-2t} - e^{-5t}\right)$$