4 STABILITY ANALYSIS

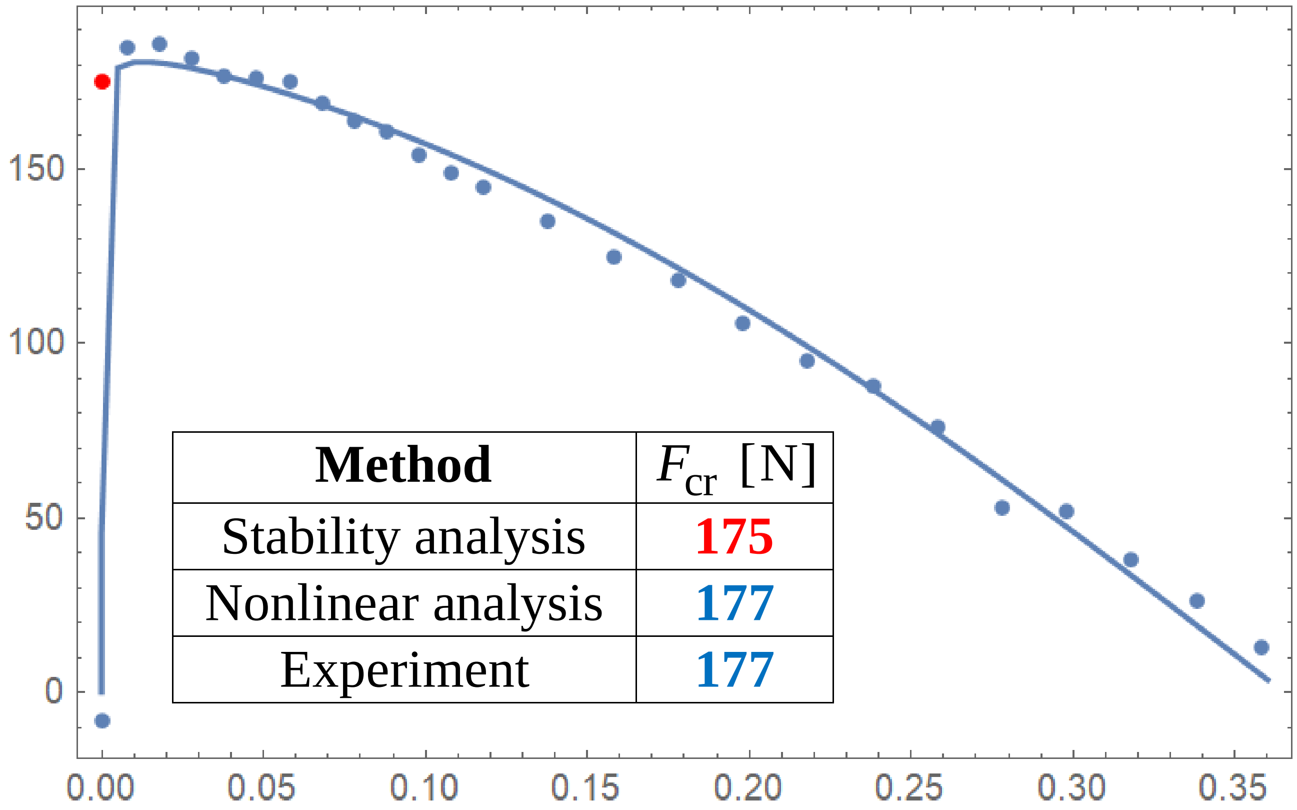
LEARNING OUTCOMES

Students are able to solve the weekly lecture problems, home problems, and exercise problems about stability FEA:

- ☐ Stability of structures and principle of virtual work for large displacements
- ☐ Aim of stability analysis and stability FEA
- ☐ Beam and plate element contributions for stability analysis

BUCKLING EXPERIMENT

EXPERIMENT VS. MODEL



Method	F_{cr} [N]
Stability analysis	175
Nonlinear analysis	177
Experiment	177

Horizontal disp. [m]

BALANCE LAWS OF MECHANICS

Balance of mass (def. of a body or a material volume) Mass of a body is constant

Balance of linear momentum (Newton 2) The rate of change of linear momentum within a material volume equals the external force resultant acting on the material volume. 🡸

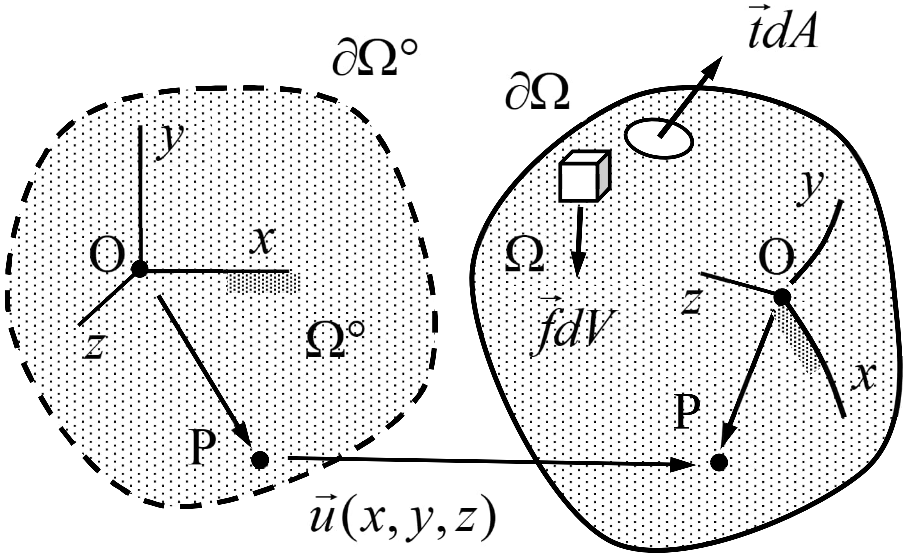
Balance of angular momentum (Cor. of Newton 2) The rate of change of angular momentum within a material volume equals the external moment resultant acting on the material volume. 🡸

Balance of energy (Thermodynamics 1)

Entropy growth (Thermodynamics 2)

INITIAL AND DEFORMED DOMAINS

Assuming equilibrium on the initial domain Ω°, the aim is to find a new equilibrium on the deformed domain Ω, when e.g., external forces acting on the structure are changed.



The local forms of the balance laws are concerned with the deformed domain which depends on the displacement! Precise treatment of large displacements requires modifications in stress and strain concepts of linear theory.

4.1 STRAIN MEASURES

A rigid body motion should not induce strains! A proper strain measure with this respect is always non-linear in displacement components (small strain $|h - h^\circ| \ll h^\circ$)

Linear strain $\varepsilon = \dfrac{h}{h^\circ} - 1 \quad \Rightarrow \quad 2\overleftrightarrow{\varepsilon} = \nabla\vec{u} + (\nabla\vec{u})_c$ epsilon

Green-Lagrange $\mathrm{E} = \dfrac{1}{2}[(\dfrac{h}{h^\circ})^2 - 1] \quad \Rightarrow \quad 2\overleftrightarrow{\mathrm{E}} = \nabla\vec{u} + (\nabla\vec{u})_c + \nabla\vec{u} \cdot (\nabla\vec{u})_c$ capital epsilon

Superscript ° refers to the initial geometry and subscript c denotes conjugate tensor. At the initial geometry, material coordinate system is usually assumed to be Cartesian so that $\nabla = \vec{i}\partial / \partial x + \vec{j}\partial / \partial y + \vec{k}\partial / \partial z$.

GENERALIZED HOOKE'S LAW

Under small displacement assumption, the model for an isotropic homogeneous material can be expressed as

Strain-stress: $$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix} = [E]^{-1} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{Bmatrix} \text{ and } \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \frac{1}{2G} \begin{Bmatrix} \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{Bmatrix}$$

Strain-displacement: $$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} = \begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial y \\ \partial u_z / \partial z \end{Bmatrix} \text{ and } \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} = \frac{1}{2} \begin{Bmatrix} \partial u_x / \partial y + \partial u_y / \partial x \\ \partial u_y / \partial z + \partial u_z / \partial y \\ \partial u_z / \partial x + \partial u_x / \partial z \end{Bmatrix}$$

Above, E is the Young's modulus, ν the Poisson's ratio, and $G = E/(2+2\nu)$ the shear modulus. Strain and stress are assumed to be symmetric.

GREEN-LAGRANGE STRAIN

A rigid body motion should not induce strains! The proper strain measures with this respect are non-linear in displacement components

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ \mathrm{E}_{zz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial u_x / \partial x)^2 + (\partial u_y / \partial x)^2 + (\partial u_z / \partial x)^2 \\ (\partial u_x / \partial y)^2 + (\partial u_y / \partial y)^2 + (\partial u_z / \partial y)^2 \\ (\partial u_x / \partial z)^2 + (\partial u_y / \partial z)^2 + (\partial u_z / \partial z)^2 \end{Bmatrix},$$

$$\begin{Bmatrix} \mathrm{E}_{xy} \\ \mathrm{E}_{yz} \\ \mathrm{E}_{zx} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial u_x / \partial x)(\partial u_x / \partial y) + (\partial u_y / \partial x)(\partial u_y / \partial y) + (\partial u_z / \partial x)(\partial u_z / \partial y) \\ (\partial u_x / \partial y)(\partial u_x / \partial z) + (\partial u_y / \partial y)(\partial u_y / \partial z) + (\partial u_z / \partial y)(\partial u_z / \partial z) \\ (\partial u_x / \partial z)(\partial u_x / \partial x) + (\partial u_y / \partial z)(\partial u_y / \partial x) + (\partial u_z / \partial z)(\partial u_z / \partial x) \end{Bmatrix}.$$

All measures boil down to the definition of linear displacement analysis when strains and rotations of material elements are small!

EXAMPLE. Consider a bar whose left end is simply supported (joint) and right end is free to move. Displacement of the typical particle (x, y) of the bar

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \begin{bmatrix} (1+\varepsilon)\cos\alpha - 1 & -\sin\alpha \\ (1+\varepsilon)\sin\alpha & \cos\alpha - 1 \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

describes rotation with angle α and length increase $\Delta h = \varepsilon h$. Determine the linear strain component ε_{xx} and the Green-Lagrange strain component E_{xx}.

Answer $\mathrm{E}_{xx} = \varepsilon + \frac{1}{2}\varepsilon^2 \approx \varepsilon$ when $|\varepsilon| \ll 1$ and $\varepsilon_{xx} = (1+\varepsilon)\cos\alpha - 1 \approx \varepsilon$ when $|\alpha| \ll 1$

- Partial derivatives of the displacement components are

$$\begin{Bmatrix} \partial u_x / \partial x \\ \partial u_y / \partial x \end{Bmatrix} = \begin{Bmatrix} (1+\varepsilon)\cos\alpha - 1 \\ (1+\varepsilon)\sin\alpha \end{Bmatrix} \text{ and } \begin{Bmatrix} \partial u_x / \partial y \\ \partial u_y / \partial y \end{Bmatrix} = \begin{Bmatrix} -\sin\alpha \\ \cos\alpha - 1 \end{Bmatrix}.$$

- Linear and Green-Lagrange axial strain components

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = (1+\varepsilon)\cos\alpha - 1 \text{ and } \mathrm{E}_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}(\frac{\partial u_x}{\partial x})^2 + \frac{1}{2}(\frac{\partial u_y}{\partial x})^2 = \varepsilon + \frac{1}{2}\varepsilon^2. \quad \Leftarrow$$

The former depends strongly on the rotation angle even when ε is small although pure rotation should not cause any strains. The latter does not depend on the rotation at all. Also, for small length changes, the Green-Lagrange strain is close to the relative change of length $\varepsilon = \Delta h / h^\circ$.

ELASTIC MATERIAL

Under the assumption of large displacements and small strains the Green-Lagrange strain measure does not differ much from the linear setting with small displacements and small strains. Constitutive equations

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ \mathrm{E}_{zz} \end{Bmatrix} = \frac{1}{C}\begin{bmatrix} 1 & -\nu & -\nu \\ -\nu & 1 & -\nu \\ -\nu & -\nu & 1 \end{bmatrix}\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \end{Bmatrix} \text{ and } \begin{Bmatrix} \mathrm{E}_{xy} \\ \mathrm{E}_{yz} \\ \mathrm{E}_{zx} \end{Bmatrix} = \frac{1}{2G}\begin{Bmatrix} S_{xy} \\ S_{yz} \\ S_{zx} \end{Bmatrix},$$

with material parameters C (which replaces E), ν, and $G = C/(2+2\nu)$ are same as those of the linear case, are assumed to simplify the setting. Also, the uni-axial and two-axial (plane) stress and strain relationships follows just by using strains instead of engineering strains and C instead of E.

STRAIN COMPONENTS FOR BUCKLING ANALYSIS

In buckling analysis of beams and plates, the setting is simplified by using the displacement assumptions of the small displacement theory and only the most significant terms of the Green-Lagrange axial strain expressions:

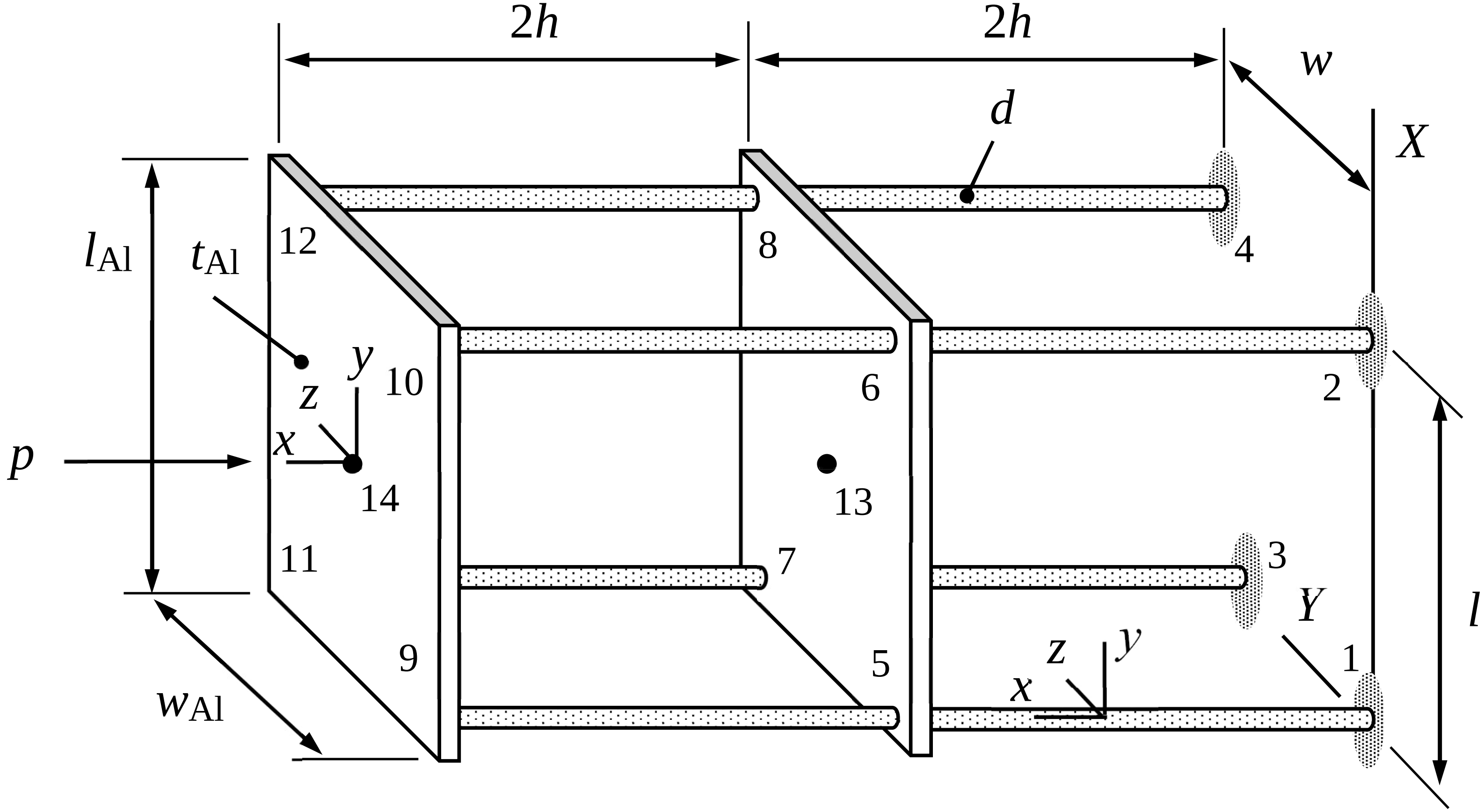
Beam: $\mathrm{E}_{xx} \approx \varepsilon_{xx} + \frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2$ and $S_{xx} = C\mathrm{E}_{xx}$,

Plate: $\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} \approx \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial w / \partial x)^2 \\ (\partial w / \partial y)^2 \\ 2(\partial w / \partial x)(\partial w / \partial y) \end{Bmatrix}$ and $\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{xy} \end{Bmatrix} = [E]_{\sigma} \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}$.

In large displacement theory, also the displacement assumptions need to be modified to keep the idea of rigid body motion of cross-sections (beams) or line segments (plates).

4.2 BUCKLING OF BEAMS AND PLATES

In stability analysis, the goal is to find the critical value p_{cr} of parameter p (force, load, displacement etc.) so that the zero and non-zero bending solutions may co-exist.



NON-LINEAR COUPLING OF THE MODES

Buckling analysis considers the coupling of the bar/ thin-slab and bending modes. There, the bending mode is affected by the bar/thin slab mode but not the other way round. Equilibrium equations for the Bernoulli beam model and Kirchhoff plate model bending modes change to

$$EI\frac{d^4w}{dx^4} - N\frac{d^2w}{dx^2} = 0 \quad x \in \Omega,$$

Non-linear coupling of the thin slab and plate bending modes

$$D\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right) - N_{xx}\frac{\partial^2 w}{\partial x^2} - 2N_{xy}\frac{\partial^2 w}{\partial x \partial y} - N_{yy}\frac{\partial^2 w}{\partial y^2} = 0 \quad (x, y) \in \Omega,$$

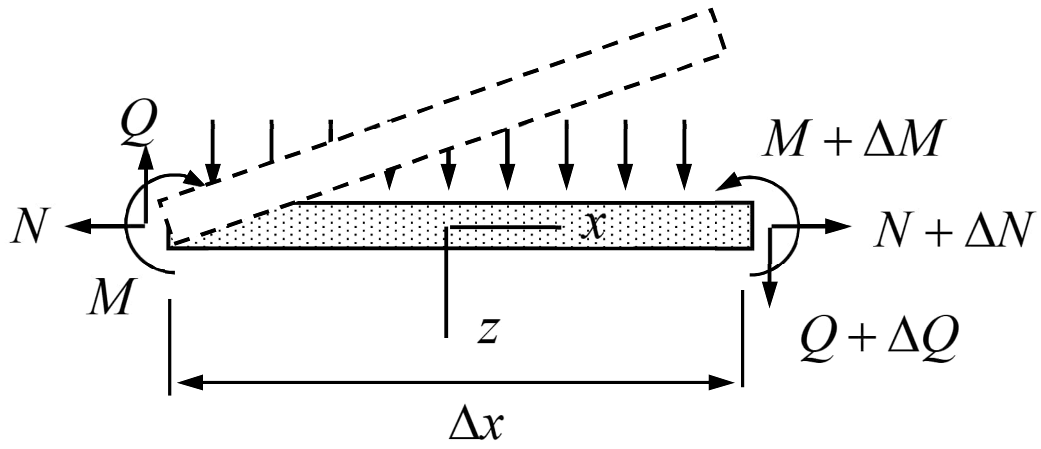
assuming that the axial or in-plane stress resultants of the bar mode or thin-slab mode are constants (as one of the assumptions).

- The simplified buckling analysis also considers the effect of the normal force on bending. By considering the equilibrium of a beam element in $xz-$plane

$$\frac{dN}{dx}=0 \quad x\in]0,L[,$$

$$\frac{dM}{dx}-Q+N\frac{dw}{dx}=0 \quad x\in]0,L[,$$

$$\frac{dQ}{dx}+f_z=0 \quad x\in]0,L[,$$



where $M=-EId^2w/dx^2$ and $N=EAdu/dx$. The more precise equilibrium equations couple the bar and bending modes (bending mode is affected by the bar mode but not the other way around).

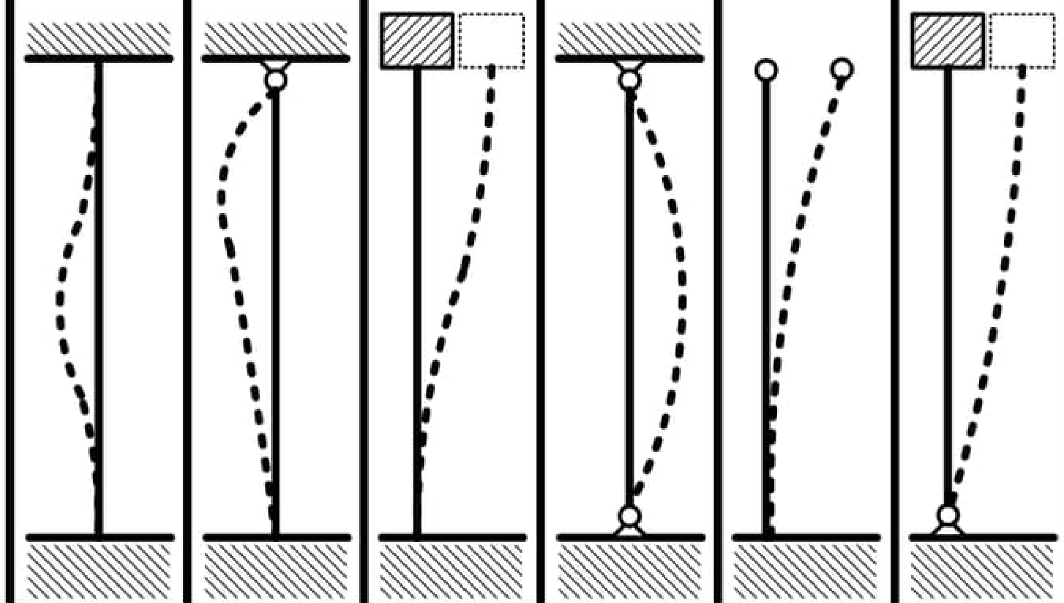
- The table by George William Herbert - *Own work, after Table C.1.8.1 in Steel Construction Manual, 8th edition, 2nd revised printing, American Institute of Steel Construction, 1987, CC BY-SA 2.5*, is based on the equilibrium equation

$$-EI\frac{d^4w}{dx^4}+N\frac{d^2w}{dx^2}=0 \quad x\in]0,L[\,,$$

for the $xz-$plane bending with a compressive $N=-p$. The different values in the table are due to different boundary and symmetry conditions imposed on the generic solution

$$w=a+bx+c\sin(\sqrt{\frac{p}{EI}}x)+d\cos(\sqrt{\frac{p}{EI}}x).$$

BUCKLING LOAD OF BEAM $p_{cr} = \pi^2 \dfrac{EI}{(KL)^2}$

Buckled shape of column shown by dashed line						
Theoretical K value	0.5	0.7	1.0	1.0	2.0	2.0
Recommended design value K	0.65	0.80	1.2	1.0	2.10	2.0

VIRTUAL WORK DENSITIES

The refined virtual work densities contain also the work done by the axial force in bending. The simplified forms of Green-Lagrange strains in derivation of virtual work densities give additional contributions (coupling terms)

Beam: $\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} d\delta v / dx \\ d\delta w / dx \end{Bmatrix}^{\mathrm{T}} N \begin{Bmatrix} dv / dx \\ dw / dx \end{Bmatrix}$ where $N = EA\dfrac{du}{dx}$,

Plate: $\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} \partial\delta w / \partial x \\ \partial\delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{xy} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix}$, $\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}$.

Coupling affects the bending mode only as the variations are concerned with the transverse displacements of the bending modes.

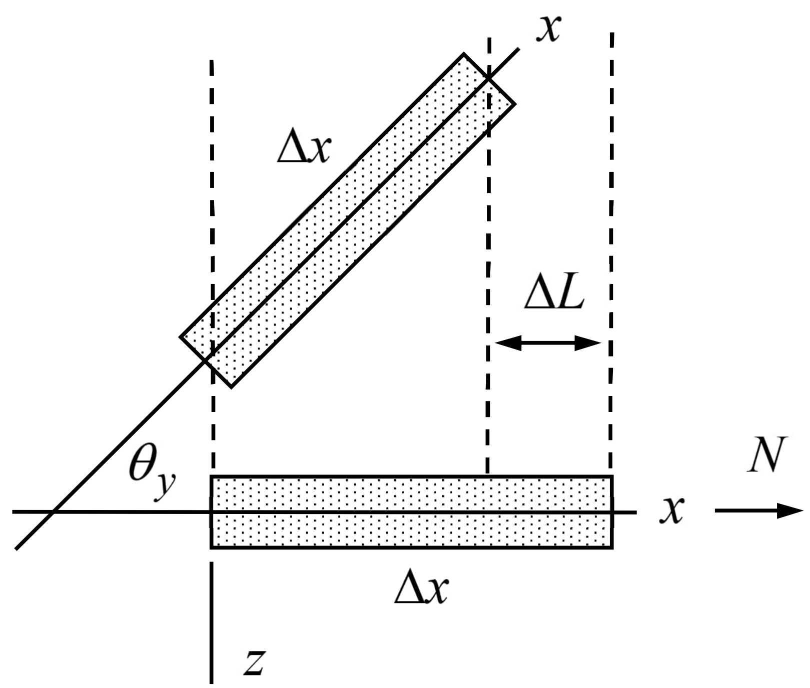
- Derivation based on the virtual work of the external axial force, is also possible. The axial displacement of the free end of a cantilever *due to the bending only* can be obtained by considering an inextensible material element of length Δx. The length change in the direction of the force is given by (Taylor series $\cos(x) = 1 - x^2/2 + \ldots$)

$$\Delta L = \Delta x - \Delta x \cos\theta_y \quad \Rightarrow$$

$$\frac{dL}{dx} = 1 - \cos\theta_y \approx \frac{1}{2}\theta_y^2 = \frac{1}{2}(-\frac{dw}{dx})^2 \quad \Rightarrow$$

$$u(L) = -\int_0^L \frac{1}{2}(\frac{dw}{dx})^2 dx \quad \Rightarrow$$

$$\delta u(L) = -\int_0^L \frac{d\delta w}{dx}\frac{dw}{dx} dx.$$



- Virtual work of the external force due to the bending effect is therefore given by

$$\delta W^{\text{sta}} = N\delta u(L) = N\int_0^L \frac{d\delta w}{dx}\frac{dw}{dx}dx.$$

- In the simultaneous bending in both directions, the length change of an inextensible material element Δx in the axial direction is given by

$$\Delta L = \Delta x - \Delta x \cos\theta_y \cos\theta_z \approx \Delta x - \Delta x(1-\frac{1}{2}\theta_y^2)(1-\frac{1}{2}\theta_z^2) \approx \Delta x\frac{1}{2}(\theta_y^2+\theta_z^2) \Rightarrow$$

$$\Delta L \approx \Delta x\frac{1}{2}(\frac{dw}{dx}\frac{dw}{dx}+\frac{dv}{dx}\frac{dv}{dx}) \Rightarrow \delta u(L) = -\int_0^L (\frac{d\delta w}{dx}\frac{dw}{dx}+\frac{d\delta v}{dx}\frac{dv}{dx})dx$$

Hence, the coupling term is the sum of coupling terms of the planar problems!

4.3 STABILITY FEA

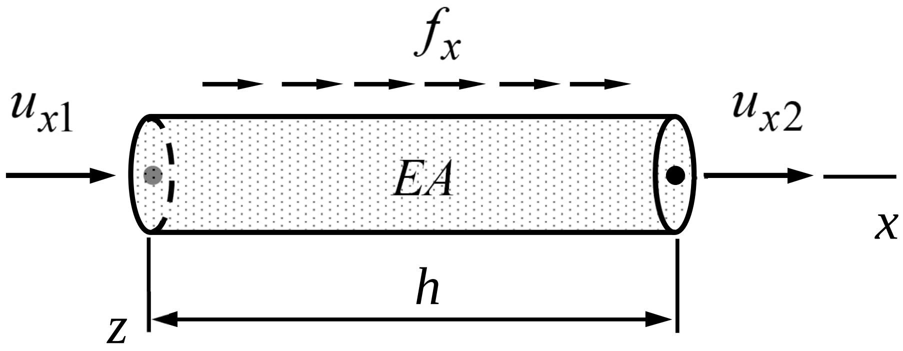
- Model a structure as a collection of beam, plate, etc. elements. Derive the element contributions $\delta W^e = \delta W^{\text{int}} + \delta W^{\text{ext}} + \delta W^{\text{sta}}$ in terms of the nodal displacement and rotation components of the structural coordinate system.

- Sum the element contributions to end up with the virtual work expression of the structure $\delta W = \sum_{e \in E} \delta W^e$. Re-arrange to get $\delta W = -\delta \mathbf{a}^{\mathrm{T}} \mathbf{R}(\mathbf{a})$.

- Use the principle of virtual work $\delta W = 0 \ \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus for $\delta \mathbf{a} \in \mathbb{R}^n$ to deduce the equilibrium equations $\mathbf{R}(\mathbf{a}) = 0$. Finally, find the values of the loading parameter p making the solution non-unique. In practice, solve for the bar/thin slab modes from the linear part and use the solution to express the axial and in-plane stress resultants of the non-linear terms in terms of p.

BAR MODE

In terms of the nodal axial forces N_{x1}, N_{x2} and nodal displacements u_{x1}, u_{x2} virtual work expressions of the internal and external forces take the forms

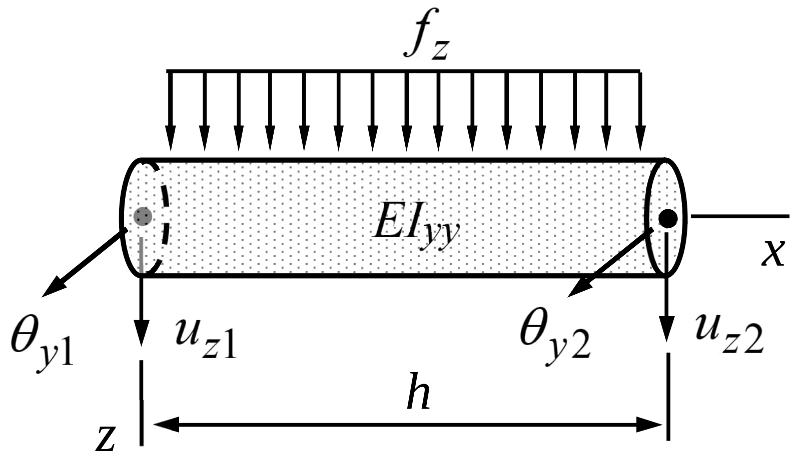


$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \begin{Bmatrix} N_{x1} \\ N_{x2} \end{Bmatrix} \quad \text{where} \quad \begin{Bmatrix} N_{x1} \\ N_{x2} \end{Bmatrix} = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_{x1} \\ u_{x2} \end{Bmatrix},$$

$$\delta W^{\text{ext}} = \begin{Bmatrix} \delta u_{x1} \\ \delta u_{x2} \end{Bmatrix}^{\text{T}} \frac{f_x h}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}.$$

BENDING MODE (*xz*-plane)

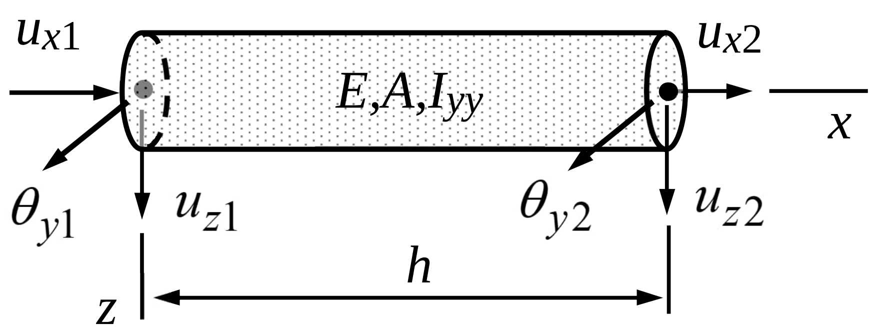
In terms of the shear forces Q_{z1}, Q_{z2}, bending moments M_{y1}, M_{y2}, displacements u_{x1}, transverse displacements u_{z1}, u_{z2}, and rotations θ_{y1}, θ_{y2}, virtual work expression of internal forces



$$\delta W^{\text{int}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} Q_{z1} \\ M_{y1} \\ Q_{z2} \\ M_{y2} \end{Bmatrix} \quad \text{where} \quad \begin{Bmatrix} Q_{z1} \\ M_{y1} \\ Q_{z2} \\ M_{y2} \end{Bmatrix} = \frac{EI_{yy}}{h^3} \begin{bmatrix} 12 & -6h & -12 & -6h \\ -6h & 4h^2 & 6h & 2h^2 \\ -12 & 6h & 12 & 6h \\ -6h & 2h^2 & 6h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}.$$

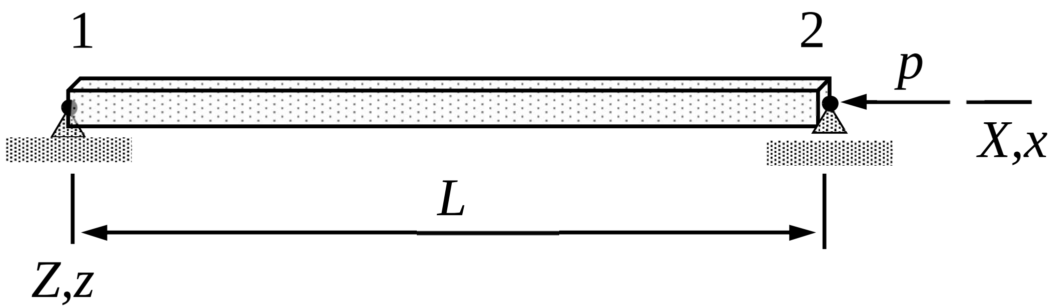
BENDING-BAR COUPLING (*xz*-plane)

Assuming a cubic approximation to $w(x)$ of nodal displacements/rotations u_{z1}, u_{z2}, θ_{y1}, and θ_{y2}, and a linear approximation to $u(x)$ of the nodal displacements u_{x1}, u_{x2}



$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h}\begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix}\begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \quad \text{where } N = EA\frac{u_{x2}-u_{x1}}{h}$$

EXAMPLE 4.1 Consider a simply supported beam loaded by a compressive axial force p acting on the right end. Assuming that displacement is confined to the $xz-$plane, use a single beam element to determine the buckling force p_{cr}. Cross-section properties A, I and Young's modulus E are constants.



Answer $p_{\text{cr}} = 12\dfrac{EI}{L^2}$ (exact to the model $p_{\text{cr}} = \pi^2\dfrac{EI}{L^2}$)

- The non-zero nodal displacements/rotations are θ_{Y1}, θ_{Y2}, and u_{X2}. Virtual work expression for the beam $\delta W^1 = \delta W^{\text{int}} + \delta W^{\text{sta}}$ and the point force δW^2 are (here $N = EA(u_{x2} - u_{x1}) / h = EAu_{X2} / L$)

$$\delta W^1 = -\delta u_{X2} \frac{EA}{L} u_{X2} - \begin{Bmatrix} \delta\theta_{Y1} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EI}{L} \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} \delta\theta_{Y1} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{NL}{30} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^2 = -p\delta u_{X2}.$$

- Virtual work expression is sum of the element contributions

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta\theta_{Y1} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} [\frac{1}{L} \begin{bmatrix} EA & 0 & 0 \\ 0 & 4EI & 2EI \\ 0 & 2EI & 4EI \end{bmatrix} + \frac{EAu_{X2}}{30} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}) \begin{Bmatrix} u_{X2} \\ \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} + \begin{Bmatrix} p \\ 0 \\ 0 \end{Bmatrix}].$$

- Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\left(\frac{1}{L}\begin{bmatrix} EA & 0 & 0 \\ 0 & 4EI & 2EI \\ 0 & 2EI & 4EI \end{bmatrix} + \frac{EAu_{X2}}{30}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} u_{X2} \\ \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} + \begin{Bmatrix} p \\ 0 \\ 0 \end{Bmatrix} = 0.$$

- The remaining task is to solve the (non-linear) equations for the values of the loading parameter p and the corresponding modes. Solving for the axial displacements (and thereby the axial forces) of the beams allowed to buckle as functions of the loading parameters is always the first step. The first equation gives

$$\frac{1}{L}EAu_{X2} + p = 0 \quad \Leftrightarrow \quad u_{X2} = -\frac{pL}{EA}.$$

- When the solution is substituted there, the remaining equations simplify to the homogeneous form

$$\left(\frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \frac{pL}{30}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right)\begin{Bmatrix} \theta_{Y1} \\ \theta_{Y2} \end{Bmatrix} = 0.$$

- A non-trivial solution (zero rotations satisfy the equations always) is possible only if the matrix in parenthesis is singular

$$\det\left(\frac{EI}{L}\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \frac{pL}{30}\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix}\right) = \left(4\frac{EI}{L} - 4\frac{pL}{30}\right)^2 - \left(2\frac{EI}{L} + \frac{pL}{30}\right)^2 = 0 \quad \Rightarrow$$

$$\frac{pL^2}{EI} \in \{12, 60\}.$$

- The smallest of the values is the critical one

$$p_{\mathrm{cr}} = 12\frac{EI}{L^2}.$$

- Stability analysis by the Mathematica code gives

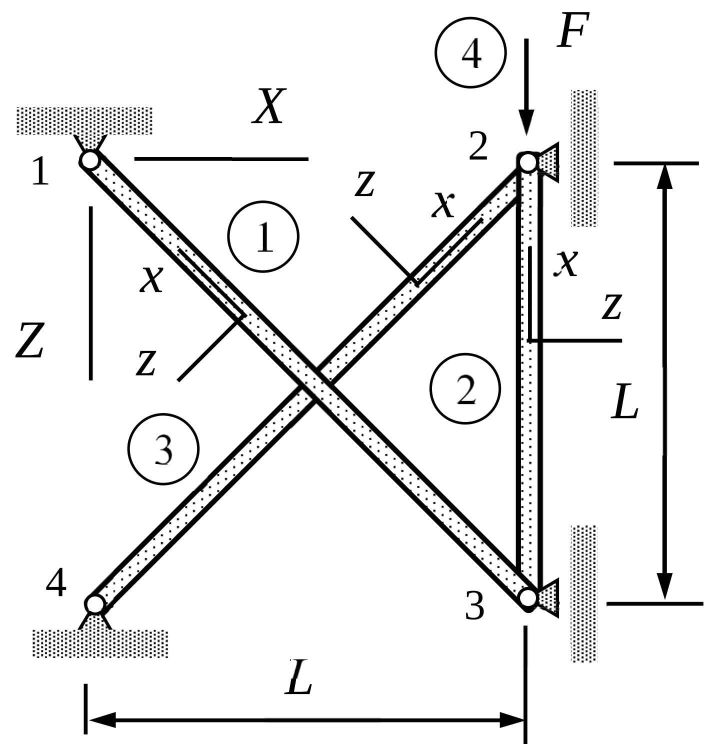
	model	properties	geometry
1	BEAM	{{E, G}, {A, I, I}}	Line[{1, 2}]
2	FORCE	{-p, 0, 0}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, θY[1], 0}
2	{L, 0, 0}	{uX[2], 0, 0}	{0, θY[2], 0}

$$p[1] \to \frac{60\,E\,I}{L^2} \quad \{uX[2] \to 0,\ \theta Y[1] \to 1,\ \theta Y[2] \to 1\}$$

$$p[2] \to \frac{12\,E\,I}{L^2} \quad \{uX[2] \to 0,\ \theta Y[1] \to -1,\ \theta Y[2] \to 1\}$$

EXAMPLE 4.2 Consider the truss shown in which elements 1 and 3 are modelled as bars and element 2 as a beam. Determine the critical value of force F for buckling of the beam element. Cross-sectional area of element 1 and 3 are $\sqrt{8}A$. Cross sectional area of element 2 is A and the second moment of area I. Young's modulus of the material is E. Assume that $\theta_{Y3} = -\theta_{Y2}$.



Answer $F_{cr} = 36\frac{EI}{L^2}$

- The non-zero nodal displacements/rotations are θ_{Y2}, $\theta_{Y3} = -\theta_{Y2}$, u_{Z2}, and u_{Z3}. Virtual work expressions of the elements are (here the axial force is given by $N = EA(u_{x2} - u_{x3})/L = EA(u_{Z3} - u_{Z2})/L$)

$$\delta W^1 = -\begin{Bmatrix} -\delta u_{Z3} \\ 0 \end{Bmatrix}^{\mathrm{T}} \frac{E\sqrt{8}A}{\sqrt{8}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} -u_{Z3} \\ 0 \end{Bmatrix} = \begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^2 = -\begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} (\frac{1}{L}\begin{bmatrix} EA & -EA & 0 \\ -EA & EA & 0 \\ 0 & 0 & 4EI + NL^2/3 \end{bmatrix}\begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^3 = -\begin{Bmatrix} 0 \\ -\delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \frac{E\sqrt{8}A}{\sqrt{8}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ -u_{Z2} \end{Bmatrix} = \begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta\theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \frac{EA}{L}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix},$$

$$\delta W^4 = \delta u_{Z2} F = \begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta \theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix}.$$

- Virtual work expression is the sum of element contributions

$$\delta W = -\begin{Bmatrix} \delta u_{Z3} \\ \delta u_{Z2} \\ \delta \theta_{Y2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{1}{L}\begin{bmatrix} 2EA & -EA & 0 \\ -EA & 2EA & 0 \\ 0 & 0 & 4EI + NL^2/3 \end{bmatrix} \begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix}\right).$$

- Principle of virtual work and the fundamental lemma of variation calculus imply that

$$\frac{1}{L}\begin{bmatrix} 2EA & -EA & 0 \\ -EA & 2EA & 0 \\ 0 & 0 & 4EI + NL^2/3 \end{bmatrix} \begin{Bmatrix} u_{Z3} \\ u_{Z2} \\ \theta_{Y2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \\ 0 \end{Bmatrix} = 0 \quad \text{where} \quad N = \frac{EA}{L}(u_{Z3} - u_{Z2}).$$

- The remaining task is to solve the (non-linear) equations for the values of the loading parameter F making the solution non-unique (the corresponding modes might be of some interest also). The first two equations give

$$\frac{EA}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} u_{Z3} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{Z3} \\ u_{Z2} \end{Bmatrix} = \frac{FL}{3EA}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}.$$

- When the solution is substituted there, the axial force expression and the remaining third equation give

$$N = \frac{EA}{L}(u_{Z3} - u_{Z2}) = -\frac{F}{3} \quad \Rightarrow \quad (4EI - \frac{1}{3}\frac{FL^2}{3})\theta_{Y2} = 0 \ .$$

- A non-trivial solution $\theta_{Y2} \neq 0$ is possible only if

$$4EI - \frac{FL^2}{9} = 0 \quad \Leftrightarrow \quad F_{\text{cr}} = 36\frac{EI}{L^2}\ .$$

- Stability analysis by the Mathematica code gives

	model	properties	geometry
1	BAR	$\{\{E\}, \{2\sqrt{2}\,A\}\}$	Line[{1, 3}]
2	BEAM	{{E, G}, {A, I, I}}	Line[{2, 3}]
3	BAR	$\{\{E\}, \{2\sqrt{2}\,A\}\}$	Line[{4, 2}]
4	FORCE	{0, 0, F}	Point[{2}]

	{X,Y,Z}	$\{u_X, u_Y, u_Z\}$	$\{\theta_X, \theta_Y, \theta_Z\}$
1	{0, 0, 0}	{0, 0, 0}	{0, 0, 0}
2	{L, 0, 0}	{0, 0, uZ[2]}	{0, θY[2], 0}
3	{L, 0, L}	{0, 0, uZ[3]}	{0, −θY[2], 0}
4	{0, 0, L}	{0, 0, 0}	{0, 0, 0}

$$F[1] \to \frac{36\,E\,I}{L^2} \quad \{uZ[2] \to 0,\ uZ[3] \to 0,\ \theta Y[2] \to 1\}$$

4.4 ELEMENT CONTRIBUTIONS

Virtual work expressions for the beam and plate elements combine virtual work densities of the model and approximation depending on the element shape and type. To derive the expression:

- Start with the virtual work densities $\delta w_{\Omega}^{\text{int}}$, $\delta w_{\Omega}^{\text{sta}}$, and $\delta w_{\Omega}^{\text{ext}}$ of the formulae collection.
- Represent the unknown functions by interpolation of the nodal displacement and rotations (see formulae collection). Substitute the approximations into the density expressions.
- Integrate the virtual work density over the domain occupied by the element to get δW .

ELEMENT APPROXIMATION

In MEC-E8001 element approximation is a polynomial interpolant of the nodal displacement and rotations in terms of shape functions. In stability analysis, shape functions depend on *x*, *y*, and *z*.

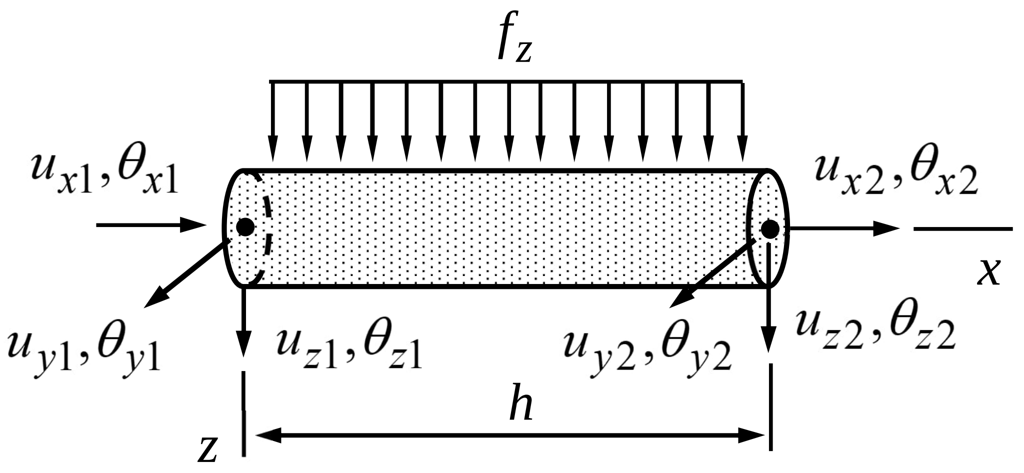
Approximation $\mathbf{u} = \mathbf{N}^{\mathrm{T}}\mathbf{a}$ always of the same form!

Shape functions $\mathbf{N} = \{N_1(x, y, z) \quad N_2(x, y, z) \quad \dots \quad N_n(x, y, z)\}^{\mathrm{T}}$

Parameters $\mathbf{a} = \{\mathrm{a}_1 \quad \mathrm{a}_2 \quad \dots \quad \mathrm{a}_n\}^{\mathrm{T}}$

Nodal parameters $\mathrm{a} \in \{u_x, u_y, u_z, \theta_x, \theta_y, \theta_z\}$ may be just displacement or rotation components or a mixture of them (as with the beam model).

BEAM MODEL



Coupling term: $\delta w_{\Omega}^{\text{sta}} = -\dfrac{d\delta v}{dx} N \dfrac{dv}{dx} - \dfrac{d\delta w}{dx} N \dfrac{dw}{dx}$, where $N = EA\dfrac{du}{dx}$.

The additional coupling term is part of the virtual work density of internal forces $\delta w_{\Omega} = (\delta w_{\Omega}^{\text{int}} + \delta w_{\Omega}^{\text{sta}}) + \delta w_{\Omega}^{\text{ext}}$ and assumes that $S_y = S_z = I_{yz} = 0$. The coupling of the bar and bending modes is the most significant non-linear term.

- The coupling terms of the bending and bar modes follow from the large displacement virtual work expression and displacement assumptions. For the beam model $u_x = u - zdw/dx - ydv/dx$, $u_y = v(x)$, and $u_z = w(x)$. Considering only the most significant terms of the Green-Lagrange axial strain expression

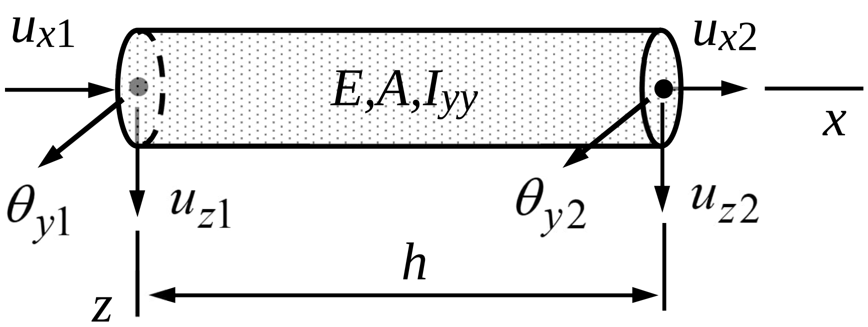
$$E_{xx} = \frac{du}{dx} - z\frac{d^2w}{dx^2} - y\frac{d^2v}{dx^2} + \frac{1}{2}(\frac{dv}{dx})^2 + \frac{1}{2}(\frac{dw}{dx})^2 \text{ and } S_{xx} = CE_{xx},$$

- Integration of $\delta w^{\text{int}} = -\delta E_{xx} S_{xx}$ over the cross-section gives the virtual work densities of the bar mode, bending modes, and the additional coupling term. Assuming that $S_y = S_z = I_{yz} = 0$, the additional coupling term takes the form

$$\delta w_{\Omega}^{\text{sta}} = -\frac{d\delta v}{dx} N \frac{dv}{dx} - \frac{d\delta w}{dx} N \frac{dw}{dx}, \text{ where } N = EA\frac{du}{dx}.$$

BENDING-BAR COUPLING (*xz*-plane)

Assuming that $v = 0$, $\phi = 0$, a cubic approximation to $w(x)$ in terms of nodal displacements/rotations u_{z1}, u_{z2}, θ_{y1}, and θ_{y2}, and a linear approximation to $u(x)$ in terms of the nodal displacements u_{x1}, u_{x2},



$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \text{ where } N = EA\frac{u_{x2} - u_{x1}}{h}.$$

- Virtual work density of the bending-bar mode coupling term in the $xz-$plane is given by

$$\delta w_{\Omega}^{\mathrm{sta}} = -N \frac{d\delta w}{dx}\frac{dw}{dx} \quad \text{where } N = EA\frac{du}{dx}$$

and the cross-sectional area A and Young's modulus E may depend on x. Element approximations (simplest possible) are $du/dx = (u_{x2} - u_{x1})/h$ and

$$w = \frac{1}{h^3}\begin{Bmatrix} (h-x)^2(h+2x) \\ -h(h-x)^2 x \\ (3h-2x)x^2 \\ (h-x)x^2 \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix} \Rightarrow \frac{dw}{dx} = \frac{1}{h^3}\begin{Bmatrix} -6(h-x)x \\ -h(h-3x)(h-x) \\ 6(h-x)x \\ h(2h-3x)x \end{Bmatrix}^{\mathrm{T}} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}.$$

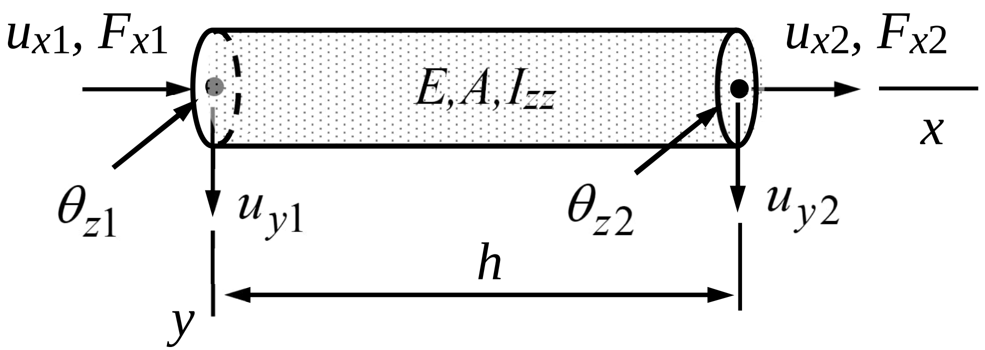
- Integration over the domain occupied by the element gives

$$\delta W^{\text{sta}} = \int_0^h \delta w_\Omega^{\text{sta}} dx = -N \int_0^h \frac{d\delta w}{dx}\frac{dw}{dx} dx \quad \left(N = EA\frac{du}{dx} \text{ is constant here}\right) \quad \Rightarrow$$

$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{z1} \\ \delta\theta_{y1} \\ \delta u_{z2} \\ \delta\theta_{y2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & -3h & -36 & -3h \\ -3h & 4h^2 & 3h & -h^2 \\ -36 & 3h & 36 & 3h \\ -3h & -h^2 & 3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{z1} \\ \theta_{y1} \\ u_{z2} \\ \theta_{y2} \end{Bmatrix}, \text{ where } N = EA\frac{u_{x2} - u_{x1}}{h}. \;\Leftarrow$$

BENDING-BAR COUPLING (*xy*-plane)

Assuming a cubic approximation to $v(x)$ in terms of nodal displacements/rotations u_{y1}, u_{y2}, θ_{z1}, and θ_{z2}, and linear approximation to $u(x)$ in terms of nodal displacements u_{x1}, u_{x2},



$$\delta W^{\text{sta}} = -\begin{Bmatrix} \delta u_{y1} \\ \delta\theta_{z1} \\ \delta u_{y2} \\ \delta\theta_{z2} \end{Bmatrix}^{\mathrm{T}} \frac{N}{30h} \begin{bmatrix} 36 & 3h & -36 & 3h \\ 3h & 4h^2 & -3h & -h^2 \\ -36 & -3h & 36 & -3h \\ 3h & -h^2 & -3h & 4h^2 \end{bmatrix} \begin{Bmatrix} u_{y1} \\ \theta_{z1} \\ u_{y2} \\ \theta_{z2} \end{Bmatrix}, \quad \text{where } N = EA\frac{u_{x2} - u_{x1}}{h}.$$

PLATE MODEL

Virtual work density combines the thin-slab and plate bending modes. Assuming that the material coordinate system is placed at the geometric mid-plane, bending mode is affected by the thin slab mode but not vice versa. The additional coupling term for stability analysis

Coupling: $\delta w_{\Omega}^{\mathrm{sta}} = -\begin{Bmatrix} \partial\delta w / \partial x \\ \partial\delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix}$, where $\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_{\sigma} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix}$

depends on the in-plane stress resultants N_{xx}, N_{yy}, and $N_{xy} = N_{yx}$ of the thin-slab mode. The additional coupling term is part of the virtual work density of internal forces $\delta w_{\Omega} = \delta w_{\Omega}^{\mathrm{int}} + \delta w_{\Omega}^{\mathrm{sta}} + \delta w_{\Omega}^{\mathrm{ext}}$. As stability term affects only the bending mode, dependence of the stress resultants on the loading parameter can be obtained from a thin-slab problem.

- The coupling term of the plate bending and thin-slab loading modes follows from the generic non-linear virtual work density of the internal forces and the kinematic assumptions of the Kirchhoff plate model $u_x = u - z\partial w / \partial x$, $u_y = v - z\partial w / \partial y$, and $u_z = w(x, y)$. If only the most significant terms are accounted for, Green-Lagrange strain and the corresponding second Piola-Kirchhoff stress components

$$\begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix} \approx \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix} - z \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix} + \frac{1}{2} \begin{Bmatrix} (\partial w / \partial x)^2 \\ (\partial w / \partial y)^2 \\ 2(\partial w / \partial x)(\partial w / \partial y) \end{Bmatrix},$$

$$\begin{Bmatrix} S_{xx} \\ S_{yy} \\ S_{xy} \end{Bmatrix} = [E]_\sigma \begin{Bmatrix} \mathrm{E}_{xx} \\ \mathrm{E}_{yy} \\ 2\mathrm{E}_{xy} \end{Bmatrix}.$$

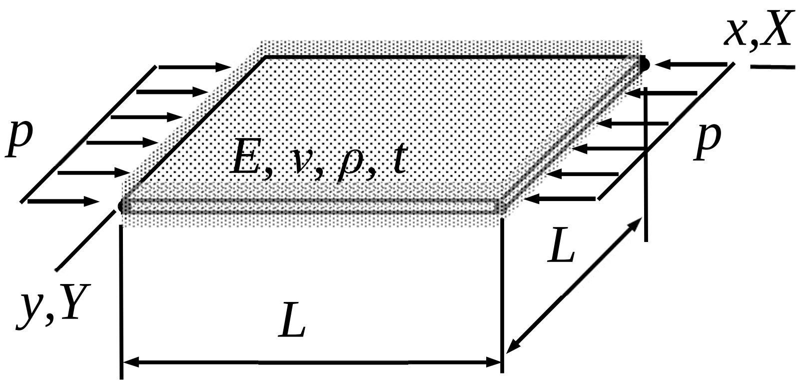
- Assuming that the material coordinate system is placed at the geometric mid-plane, integration of the virtual work density gives the virtual work density of the thin-slab mode, virtual work density of plate bending mode, and the coupling term (considering only the most significant terms)

$$\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} \partial \delta w / \partial x \\ \partial \delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix},$$

where the in-plane stress resultants

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = t[E]_{\sigma} \begin{Bmatrix} \partial u / \partial x \\ \partial v / \partial y \\ \partial u / \partial y + \partial v / \partial x \end{Bmatrix}.$$

EXAMPLE 4.3 Determine the critical value of the in-plane loading p_{cr} making the plate of the figure to buckle. Use the approximation $w(x,y) = a_0(xy/L^2)(1-x/L)(1-y/L)$. Assume that the edge conditions are such that solution to the in-plane stress resultants is given by $N_{xx} = -p$ and $N_{yy} = N_{xy} = 0$ (solution to the thin-slab problem).



Answer $p_{cr} = \dfrac{11}{3}\dfrac{Et^3}{L^2(1-\nu^2)}$ (exact $p_{cr} = \dfrac{\pi^2}{3}\dfrac{Et^3}{L^2(1-\nu^2)}$).

- Assuming that the material coordinate system is chosen so that the linear plate bending and thin slab modes decouple, the plate model virtual work densities of the bending mode and the coupling term are given by ($N_{xx} = -p$ and $N_{yy} = N_{xy} = 0$)

$$\delta w_{\Omega}^{\text{int}} = -\begin{Bmatrix} \partial^2 \delta w / \partial x^2 \\ \partial^2 \delta w / \partial y^2 \\ 2\partial^2 \delta w / \partial x \partial y \end{Bmatrix}^{\mathrm{T}} D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \partial^2 w / \partial x^2 \\ \partial^2 w / \partial y^2 \\ 2\partial^2 w / \partial x \partial y \end{Bmatrix} \text{ where } D = \frac{t^3}{12} \frac{E}{1-\nu^2},$$

$$\delta w_{\Omega}^{\text{sta}} = -\begin{Bmatrix} \partial \delta w / \partial x \\ \partial \delta w / \partial y \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} N_{xx} & N_{xy} \\ N_{yx} & N_{yy} \end{bmatrix} \begin{Bmatrix} \partial w / \partial x \\ \partial w / \partial y \end{Bmatrix} = \frac{\partial \delta w}{\partial x} p \frac{\partial w}{\partial x}.$$

- When the approximation is substituted there, virtual work expressions of the plate bending mode and that of the coupling between the thin-slab and bending modes simplify to

$$\delta W^{\text{int}} = \int_0^L \int_0^L \delta w_\Omega^{\text{int}} dxdy = -\delta a_0 \frac{22}{45}\frac{D}{L^2} a_0,$$

$$\delta W^{\text{sta}} = \int_0^L \int_0^L \delta w_\Omega^{\text{sta}} dxdy = \delta a_0 \frac{1}{90} p a_0.$$

- Virtual work expression is the sum of the two parts

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{sta}} = -\delta a_0 (\frac{22}{45}\frac{D}{L^2} - \frac{1}{90} p) a_0.$$

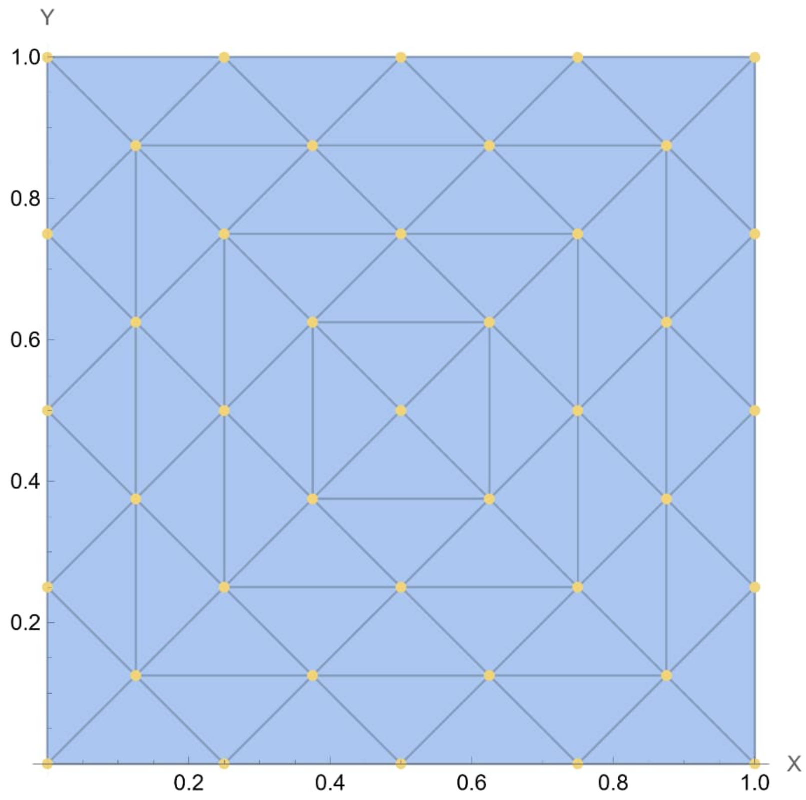
- Principle of virtual work $\delta W = 0 \, \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus give

$$\delta W = -\delta a_0 (\frac{22}{45}\frac{D}{L^2} - \frac{1}{90} p) a_0 = 0 \quad \forall \delta a_0 \quad \Rightarrow \quad (\frac{22}{45}\frac{D}{L^2} - \frac{1}{90} p) a_0 = 0.$$

For a non-trivial solution $a_0 \neq 0$, the loading parameter needs to take the value

$$p_{\mathrm{cr}} = 44\frac{D}{L^2} = \frac{11}{3}\frac{Et^3}{L^2(1-\nu^2)} \,.$$

- The problem can be solved numerically by using the Reissner-Mindlin plate model and the Mathematica code. Assuming parameter values $E = 210\,\mathrm{GPa}$, $\nu = 0.33$, $L = 1\mathrm{m}$, and $t = 1\mathrm{mm}$, the one parameter approximation gives $p_{\mathrm{cr}} = 0.77\,\mathrm{Nm}^{-1}$ whereas the solution on the mesh shown gives $p_{\mathrm{cr}} = 0.78\,\mathrm{Nm}^{-1}$.



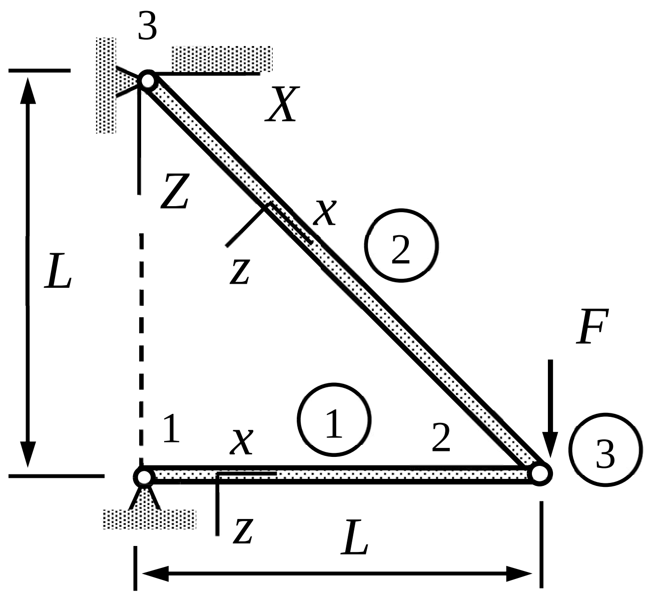
STABILITY ANALYSIS OF TRUSS SIMPLIFIED

In hand calculations, one may use the fact that the bar model predicts the axial forces correctly when beams of a truss are connected with joints. Then, the first step is a linear displacement analysis for finding the displacements of the nodes and thereby the axial forces $N(p)$ as functions of the loading parameter. After that, the buckling loads of each beam under compression follows from the buckling criterion (N is negative in compression)

$$-N(p) = \pi^2 \frac{EI}{L^2}$$

for a simply supported beam. The first beam to buckle (or the smallest p given by the conditions above) defines the critical load p_{cr}.

EXAMPLE 4.4 A beam truss is loaded by a vertical point force having magnitude F and acting in the positive or negative direction of the Z-axis. Determine the critical load magnitude F_{cr} for buckling of beam 1 or 2 of the truss. Cross-sectional area of element 1 is A and that for element 2 $\sqrt{8}A$, Young's modulus E is constant, and the second moment of area is I for both beams. The beams are connected by frictionless joints.



Answer $F_{\text{cr}} = \frac{\pi^2}{\sqrt{8}} \frac{EI}{L^2}$ when $F < 0$.

- The relationships between the nodal displacement components in the material and structural systems are $u_{x1}=0$ and $u_{x2}=u_{X2}$. Element contribution δW^1 to the virtual work expression of the structure is

$$\delta W^1 = -\begin{Bmatrix} 0 \\ \delta u_{X2} \end{Bmatrix}^{\mathrm{T}} (\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_{X2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}) = -\frac{EA}{L} u_{X2}\delta u_{X2}.$$

- For element 2, $u_{x3}=0$ and $u_{x2}=(u_{X2}+u_{Z2})/\sqrt{2}$. Element contribution takes the form

$$\delta W^2 = -\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ \delta u_{X2}+\delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} (\frac{E\sqrt{8}A}{\sqrt{2}L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 \\ u_{X2}+u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}) \quad \Leftrightarrow$$

$$\delta W^2 = -\frac{EA}{L}(\delta u_{X2}+\delta u_{Z2})(u_{X2}+u_{Z2}).$$

- Virtual work expression of the point force follows from the definition of work. The direction may be up or down and hence F may also be negative (which means up)

$$\delta W^3 = \delta u_{Z2} F \, .$$

- Virtual work expression of a structure is obtained as the sum of the element contributions

$$\delta W = -\frac{EA}{L}\delta u_{X2} u_{X2} - \frac{EA}{L}(\delta u_{X2} + \delta u_{Z2})(u_{X2} + u_{Z2}) + \delta u_{Z2} F \quad \Leftrightarrow$$

$$\delta W = -\begin{Bmatrix} \delta u_{X2} \\ \delta u_{Z2} \end{Bmatrix}^{\mathrm{T}} \left(\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} + \begin{Bmatrix} 0 \\ F \end{Bmatrix}\right).$$

- Using the principle of virtual work $\delta W = 0 \;\; \forall \delta \mathbf{a}$ and the fundamental lemma of variation calculus

$$\frac{EA}{L}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} - \begin{Bmatrix} 0 \\ F \end{Bmatrix} = 0 \quad \Leftrightarrow \quad \begin{Bmatrix} u_{X2} \\ u_{Z2} \end{Bmatrix} = \frac{LF}{EA}\begin{Bmatrix} -1 \\ 2 \end{Bmatrix}.$$

- For buckling of beam 1, the axial force should be compression (negative) and therefore the external force should be acting downwards.

$$N = \frac{EA}{L}(u_{x2} - u_{x1}) = \frac{EA}{L}u_{X2} = -F \quad \Rightarrow \quad F_{\text{cr}} = \pi^2 \frac{EI}{L^2} \text{ when } F > 0.$$

- For buckling of beam 2, the axial force should be compression (negative) and therefore the external force should be acting upwards. When $F < 0$

$$N = \frac{E\sqrt{8}A}{\sqrt{2}L}(u_{x2} - u_{x3}) = \sqrt{2}\frac{EA}{L}(u_{X2} + u_{Z2}) = -\sqrt{2}F \quad \Rightarrow \quad F_{\text{cr}} = \frac{\pi^2}{\sqrt{8}}\frac{EI}{L^2}.$$