ELEC-C1310 Automaatio- ja systeemitekniikan laboratoriotyöt

Control of an inverted pendulum

Background on the theory on Sections 3.2.3-3.2.4 in the workbook "**Rotary Pendulum Workbook (Student).pdf**".

The theory of state feedback design by "companion forms" is well-known in control theory. However, in the workbook in contains errors and is also badly written. This short document explains the technique and its background.

Let the single-input single output process be given in state-space form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

where *A, x, B, u, y* and *C* are *nxn, nx1, nx1, 1x1, 1x1* and *1xn* – dimensional matrices, respectively. This is a *representation* of the process, describing its input-output dynamics.

Consider the state variable change

$$x(t) = W\tilde{x}(t) \Rightarrow \tilde{x}(t) = W^{-1}x(t) \tag{2}$$

where *W* is any non-singular (inverse matrix exists) *nxn*-dimensional matrix. It follows (time *t* is dropped from equations when convenient and when no confusion is possible)

$$\begin{aligned} \dot{\tilde{x}} &= W^{-1}\dot{x} = W^{-1}Ax + W^{-1}Bu = \underbrace{W^{-1}AW}_{\tilde{A}}\,\tilde{x} + \underbrace{W^{-1}B}_{\tilde{B}}\,u \\ y &= Cx = \underbrace{CW}_{\tilde{C}}\,\tilde{x} \end{aligned} \tag{3}$$

We now have a new *representation* of the process

$$\begin{aligned} \dot{\tilde{x}}(t) &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t) \\ y(t) &= \tilde{C}\tilde{x}(t) \end{aligned} \tag{4}$$

so that the representations (1) and (4) are *equivalent* through the *similarity transformation* (2). They represent the same process, however, demonstrating the fact that the state-space representation of a process is not unique. In fact, by the above transformation an *infinite* number of state-space representations for the same process exist, by the change of the state variable as in (2).

Let us now check that (1) and (4) give the same transfer function (input-output behaviour) and the same characteristic polynomial. To that end, first look at two results of matrix algebra, which are used:

1. It is well-known (not proved here) that for any two square matrices of the same dimension it holds $\det(XY) = \det(X) \cdot \det(Y)$

2. For any invertible square matrix it holds $\det(X^{-1}) = 1/\det(X)$. Proof: $\det(XX^{-1}) = \det(X)\cdot\det(X^{-1}) = 1 \Rightarrow \det(X^{-1}) = 1/\det(X)$.

The transfer function of representation (1) is $G_1(s) = \dfrac{Y(s)}{U(s)} = C\left(sI - A\right)^{-1} B$. The transfer function of representation (4) is

$$G_4(s) = \frac{Y(s)}{U(s)} = \tilde{C}\left(sI - \tilde{A}\right)^{-1}\tilde{B} = CW\left(sI - W^{-1}AW\right)^{-1}W^{-1}B = CW\left[sW^{-1}W - W^{-1}AW\right]^{-1}W^{-1}B$$
$$= CW\left[W^{-1}\left(sI - A\right)W\right]^{-1}W^{-1}B = CWW^{-1}\left(sI - A\right)^{-1}WW^{-1}B = C\left(sI - A\right)^{-1}B$$

Ok. (Note that for square matrices $\left(X_1X_2\right)^{-1} = X_2^{-1}X_1^{-1}$).

The characteristic polynomial is of course the same by the transfer function result, but let us still look at it separately: The characteristic polynomial of (4) is

$$\det(sI - \tilde{A}) = \det(sI - W^{-1}AW) = \det\left(sW^{-1}W - W^{-1}AW\right) = \det\left[W^{-1}\left(sI - A\right)W\right]$$
$$= \det(W^{-1})\cdot\det(sI - A)\cdot\det(\mathrm{W}) = \frac{1}{\det(W)}\cdot\det(W)\cdot\det(sI - A) = \det(sI - A)$$

which is the same as the characteristic polynomial of (1). Note that the results 1 and 2 above were used. Also note that the determinants are scalars, so their order in multiplication can be changed freely.

It is now clear that given system (1) and any invertible transformation (2), the equivalent system (4) can be calculated. But this is not the real target. The main question is: Given (1), what possible target representations (4) can we *choose*, so that the transformation (2) exists. This is not so trivial question.

Of course, the equivalent representations (1) and (4) must have the same transfer function. But that information does not really help in calculating the matrix *W*.

Consider again the representation (1). Let the transfer function of it be

$$G_1(s) = \frac{b_n s^{n-1} + b_{n-1}s^{n-2} + \cdots + b_1}{s^n + a_n s^{n-1} + a_{n-1}s^{n-2} + \cdots + a_1} \tag{5}$$

where it has been assumed that all pole-zero-cancellations have been done. The system is the *reachable* (the student Workbook and many other sources use the term *controllable;* the two concepts are not exactly the same but are equivalent in time continuous linear time invariant systems).

The controllability matrix of (1) is

$$T = \left[B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{n-1}B\right] \tag{6}$$

and it has full rank (it is invertible), because the system is reachable (controllable). The controllability matrix of (4) is

$$\tilde{T} = \left[\tilde{B} \vdots \tilde{A}\tilde{B} \vdots \tilde{A}^2\tilde{B} \vdots \cdots \vdots \tilde{A}^{n-1}\tilde{B} \right] \tag{7}$$

(note that in the Student Workbook there is an error in this formula)

Now it is easy to see that

$$W\tilde{T} = T \tag{8}$$

Hint for proof: Note that

$$W\tilde{B} = WW^{-1}B = B$$
$$W\tilde{A}^k\tilde{B} = W\left(W^{-1}AW\right)\underbrace{(W^{-1}AW)\ \cdots\ (W^{-1}AW)}_{k\,times}W^{-1}B = A^kB$$

Now we are almost there. For the matrices in (4) take

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 & \cdots & -a_n \end{bmatrix},\ \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},\ \tilde{C} = \begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix}$$

which is called the *control canonical form* of the process given first by (1) or by the transfer function (5). (There exist other canonical forms, e.g. observable canonical form, diagonal form, Jordan form.)

It is easy to check that this form really gives the transfer function (5), which means that (4) with these matrices is really a realization (state-space representation) of the original process (1), (5). Moreover, it is easy to calculate the controllability matrix (7) and note that it has full rank (it is invertible), so it is reachable. From equation (8) we obtain the transformation *W* which transforms (1) to (4)

$$W\tilde{T} = T \Rightarrow W = T\tilde{T}^{-1} \tag{9}$$

The representations (1) and (4) are then equivalent, e.g. their poles are the same. To make a controller design for the process can be easier by using the controller canonical form because of its special matrix structure. That has been explained in Student Workbook, Section 3.2.4. For a state feedback control law we design for (4)

$$u(t) = -\tilde{K}\tilde{x}(t)$$

to get the desired closed loop poles. Finally, to control the real process we note (2) and get

$$u(t) = -\tilde{K}W^{-1}x(t)$$

which is the final control law, implemented to control the system (1).