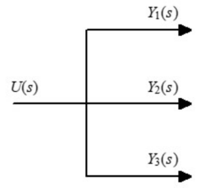
ELEC-C8201 Control and Automation

Lecture 4: Block diagram algebra, PID controller, Routh-Hurwitz stability test

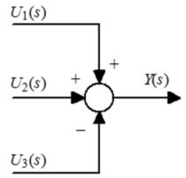
Block Diagram conversions: Signals

- In block diagrams, a single signal can be exported to more than one block (signal branching).
  - A block diagram is an information chart and it can branch information, but it does not reduce that information. Each branch has the same information.



$$Y_1(s) = Y_2(s) = Y_3(s) = U(s)$$

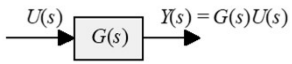
- The different signals can be combined using a summation block. The combination can be either an addition or subtraction of individual signals
  - Signs on the summation block indicate the signs of the individual signals in the total.



$$Y(s) = U_1(s) + U_2(s) - U_3(s)$$

Passing a signal through a block

- As stated in previous lectures, in Laplace domain the output signal is obtained by multiplying the input signal with a transfer function



- This basic formula can be used to derive a transformation from the serial association of the equation blocks. Introduce the auxiliary variable $\varepsilon(s)$, which is subsequently eliminated
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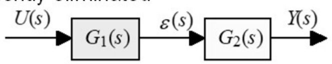
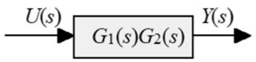


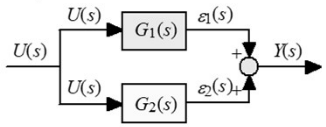
$$\begin{cases} Y(s) = G_2(s)\varepsilon(s) \\ \varepsilon(s) = G_1(s)U(s) \end{cases} \Rightarrow \quad Y(s) = G_1(s)G_2(s)U(s) = G_{TOT}(s)U(s)$$

$$\Rightarrow \quad G_{TOT}(s) = G_1(s)G_2(s) \quad \Rightarrow$$



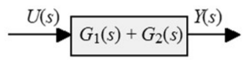
Passing a signal through a block

- The derivation for parallel blocks:



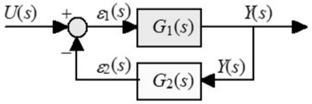
$$\begin{cases} Y(s) = \varepsilon_1(s) + \varepsilon_2(s) \\ \varepsilon_1(s) = G_1(s)U(s) \\ \varepsilon_2(s) = G_2(s)U(s) \end{cases} \Rightarrow \quad Y(s) = G_1(s)U(s) + G_2(s)U(s)$$

$$Y(s) = \left(G_1(s) + G_2(s)\right)U(s) = G_{TOT}(s)U(s) \quad \Rightarrow \quad G_{TOT}(s) = G_1(s) + G_2(s)$$



Passing a signal through a block

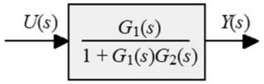
- Loop relation (feedback) to the conversion formula is calculated as:



$$\begin{cases} Y(s) = G_1(s)\varepsilon_1(s) \\ \varepsilon_1(s) = U(s) - \varepsilon_2(s) \\ \varepsilon_2(s) = G_2(s)Y(s) \end{cases} \Rightarrow \quad Y(s) = G_1(s)\big(U(s) - G_2(s)Y(s)\big)$$

$$\Rightarrow \quad Y(s) = G_1(s)U(s) - G_1(s)G_2(s)Y(s) \quad \Rightarrow \quad \big(1 + G_1(s)G_2(s)\big)Y(s) = G_1(s)U(s)$$

$$\Rightarrow \quad Y(s) = \frac{G_1(s)}{1 + G_1(s)G_2(s)}U(s) = G_{TOT}(s)U(s) \quad \Rightarrow \quad G_{TOT}(s) = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$



Thus, the overall transfer function is the tf of the forward path divided by the term (1 + open loop transfer function) This will be essential in future.

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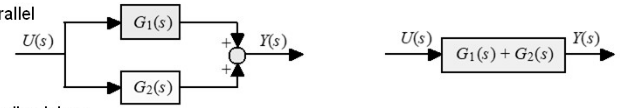
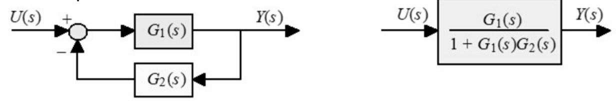
Block Diagram conversions: basic relations

- Basic block diagram relations:
  - Series

  - Parallel



  - Feedback loop



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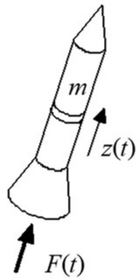
Example: Moving a rocket in space

- A rocket with a mass of M = 10 which is assumed to be constant is controlled by force F (t), which can be positive or negative. The objective is to make the location of the rocket Z (s) (1-D distance) change from the starting point (initial position) to the end point (final position).
- The position of the rocket is measured by the measuring devices with a certain amount of inertia but no bias. The transfer function of the measurement system is
- $$G_m(s) = \frac{Z_{mit}(s)}{Z(s)} = \frac{1}{\tau_m s + 1} = \frac{1}{0.1s + 1}$$
- The measured location of the rocket Z_{mit} (s) is compared to the desired location Z_{ref} (s) (i.e. reference value). The deviation between the desired position and the measured location is called an E (t) and the regulator determines the appropriate steering value for each situation. $e(t) = z_{ref}(t) - z_{mit}(t)$



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Example: Moving a rocket in space

- The controller has two parallel functions, the first of which follows the magnitude of the deviation (the difference) and multiplies it by a constant of 30 and the other with the slope of deviation (i.e. the differential derivative) and multiplies it by 31. Total control, i.e. the force required ($u(t) = F(t)$) is calculated as the sum of these two control forces.

$$\begin{cases} u_{trendi}(t) = K_D\dot{e}(t) = 31\dot{e}(t) \\ u_{poikkeama}(t) = K_P e(t) = 30e(t) \end{cases}, \quad F(t) = u(t) = u_{poikkeama}(t) + u_{trendi}(t)$$

- Suppose that the actuator is ideal (that is, the regulator directly generates the required power without inertia) so that it's inertia doesn't need to be taken into account.
- It is assumed that no damping forces are affected by the grain configuration (the force balance is limited to the inertia and thrust of the $F(t)$)
- Develop a detailed block diagram for the system and analyse the operation of both the individual blocks and the total system by means of responses.

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Example: Moving a rocket in space

- Output of rocket is a distance $\Rightarrow$ $y(t) = z(t)$
- Input to rocket is power $\Rightarrow$ $u(t) = F(t)$
- The output of the measuring device is the measured distance $\Rightarrow$ $y_{mit}(t) = z_{mit}(t)$
- The input to the measuring device is the actual distance $\Rightarrow$ $y(t) = z(t)$
- The input to the controller is $e(t)$
- The output of the controller is the power $\Rightarrow$ $u(t) = F(t)$
- Generating rocket dynamic model:
  $F(t) = m\ddot{z}(t) \Rightarrow u(t) = 10\ddot{y}(t) \Rightarrow U(s) = 10s^2Y(s) \Rightarrow G(s = \frac{Y(s)}{U(s)} = \frac{1}{10s^2}$
- The various functions of the controller are
-

$$\begin{cases} u_{trendi}(t) = 31\dot{e}(t) \\ u_{poikkeama}(t) = 30e(t) \end{cases} \Rightarrow \begin{cases} U_{trendi}(s) = 31sE(s) = G_{c1}(s)E(s) \\ U_{poikkeama}(s) = 30E(s) = G_{c2}(s)E(s) \end{cases}$$

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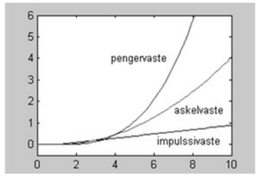
Example: Moving a rocket in space

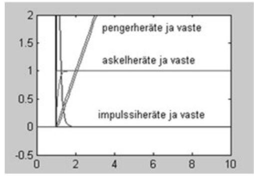
- The process (rocket) is a dual integrator with easy-to-calculate responses directly from the Laplace table.

$$\begin{cases} \text{impulse response} \quad L^{-1}\{G(s)\cdot 1\} = \frac{1}{10}t \\ \text{step response} \quad L^{-1}\{G(s)\cdot \frac{1}{s}\} = \frac{1}{20}t^2 \\ \text{ramp response} \quad L^{-1}\{G(s)\cdot \frac{1}{s^2}\} = \frac{1}{60}t^3 \end{cases}$$



- Position measurement device measures the distance correctly, but with a little lag

$$\begin{cases} \text{impulse response} \quad L^{-1}\{G_m(s)\cdot 1\} = 10e^{-10t} \\ \text{step response} \quad L^{-1}\{G_m(s)\cdot \frac{1}{s}\} = 1 - e^{-10t} \\ \text{ramp response} \quad L^{-1}\{G_m(s)\cdot \frac{1}{s^2}\} = t - \frac{1}{10}\left(1 - e^{-10t}\right) \end{cases}$$



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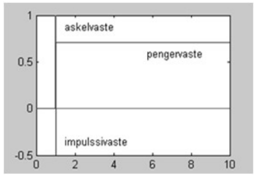
Example: Moving a rocket in space

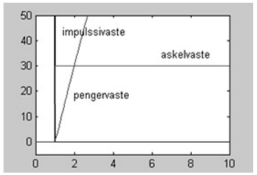
- In the differential controller, the controller reacts to the difference in the inputs

$$\begin{cases} \text{impulssivaste} \quad L^{-1}\{G_{c1}(s)\cdot 1\} = 31\dot{\delta}(t) \\ \text{askelvaste} \quad L^{-1}\left\{G_{c1}(s)\cdot \frac{1}{s}\right\} = 31\delta(t) \\ \text{pengervaste} \quad L^{-1}\left\{G_{c1}(s)\cdot \frac{1}{s^2}\right\} = 31 \end{cases}$$



- The next part of the proportional controller reacts to the current input

$$\begin{cases} \text{impulssivaste} \quad L^{-1}\{G_{c2}(s)\cdot 1\} = 30\delta(t) \\ \text{askelvaste} \quad L^{-1}\{G_{c2}(s)\cdot \frac{1}{s}\} = 30 \\ \text{pengervaste} \quad L^{-1}\{G_{c2}(s)\cdot \frac{1}{s^2}\} = 30t \end{cases}$$



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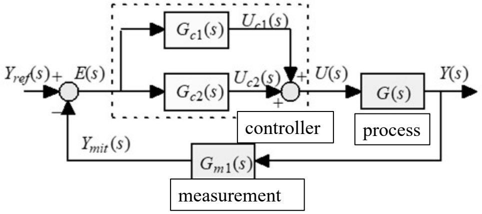
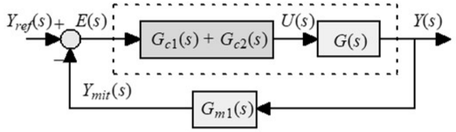
Example: Moving a rocket in space

- Now forming the entire system block diagram



- Using block chart conversions from the most innermost structure



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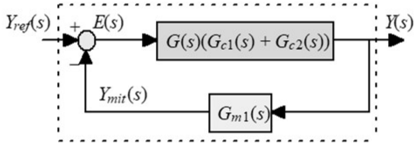
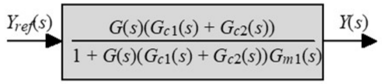
Example: Moving a rocket in space



- This concludes the entire system transfer function
-



$$Y(s) = G_{tot}(s)Y_{ref}(s) = \frac{G(s)\left(G_{c1}(s)+G_{c1}(s)\right)}{1+G(s)\left(G_{c1}(s)+G_{c1}(s)\right)G_m(s)}Y_{ref}(s) = \frac{\frac{31s+30}{10s^2}}{1+\frac{31s+30}{10s^2(0.1s+1)}}Y_{ref}(s)$$

Example: Moving a rocket in space

$$G_{tot}(s) = \frac{(31s+30)(0.1s+1)}{10s^2(0.1s+1)+31s+30} = \frac{31s^2+340s+300}{10s^3+100s^2+310s+300} = \frac{31s^2+340s+300}{10(s+2)(s+3)(s+5)}$$

- It is now possible to calculate the system response to the step of the set value, i.e. the reference to the expected position (from the moment 0 onwards, to have the position of the rocket in the desired end state).

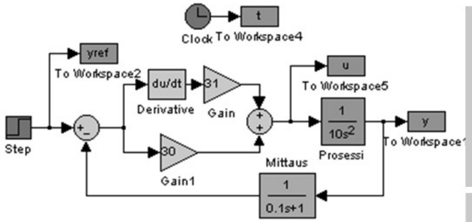
$$Y(s) = G_{tot}(s)Y_{ref}(s) = \frac{31s^2+340s+300}{10(s+2)(s+3)(s+5)}\cdot\frac{1}{s} = \frac{1}{s} + \frac{\frac{64}{15}}{s+2} - \frac{\frac{147}{30}}{s+3} + \frac{\frac{25}{12}}{s+5}$$

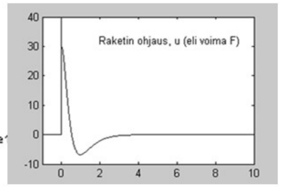
$$\Rightarrow \quad y(t) = L^{-1}\{Y(s)\} = 1 + \tfrac{64}{15}e^{-2t} - \tfrac{147}{30}e^{-3t} + \tfrac{25}{12}e^{-5t}$$

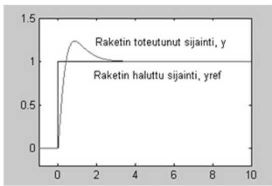
- The system works with the desire to stabilize an unstable rocket. The expression of the response approaches unity as the time approaches infinity, i.e. the desired value of distance and the actual distance approaching each other as time grows.

Example: Moving a rocket in space

- Simulating a rocket control system



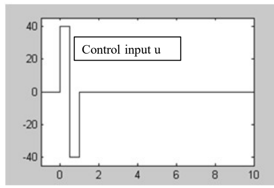


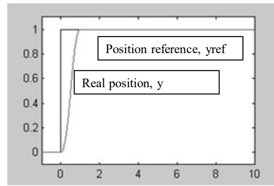


- The steering has an infinite impulse response and the step response is overshooting
  - The guidance proposed by the Controller cannot be effectively implemented and, as the distance of the rocket approaches the desired value, it will pass and overshoot (the measurement is slow)

Example: Moving a rocket in space

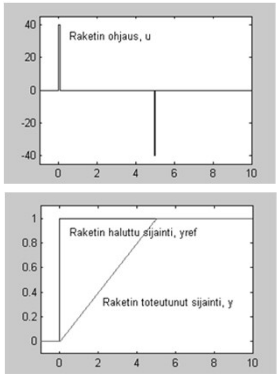
- The rocket control can also be developed with a more realistic and more effective management strategy
  - Suppose control is limited between maximum and minimum values: $u(t) \in [-40,40]$.
  - If you want to move the rocket to the desired distance in minimum time, then the solution to the optimization problem is to get the so called "Bang-bang" adjustment: that is, maximum acceleration and maximum braking
- The optimum adjustment is outside the topic of this course.





Example: Moving a rocket in space

- If you do not need to reach your destination as quickly as possible, you can optimize fuel consumption, for example (use of steering)
  - Suppose the long-term use of the steering is costly-the most economical is the fast 0.05 time unit with maximum speed.
  - The rocket must arrive only after five time units.
  - In this case, the optimization problem can be obtained: fast acceleration-smooth driving-rapid braking.



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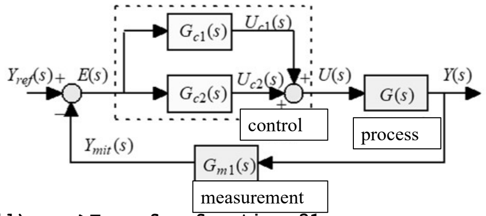
MATLAB: Block chart conversions

- Block diagram for conversions in the Control System Toolbox commands **parallel** (i.e. **+/-**), **series** (i.e. *****) and **feedback**
- Build the example of the rocket control system with a total block diagram in Matlab

$$G_m(s) = \frac{1}{0.1s+1}$$

$$G(s) = \frac{1}{10s^2}$$

$$\begin{cases} G_{c1}(s) = 31s \\ G_{c2}(s) = 30 \end{cases}$$



- `Gc1=tf([31 0],[1])` **=>Transfer function:31 s**
- `Gc2=tf([30],[1])` **=>Transfer function:30**
- `G=tf([1],[10 0 0])` **=>Transfer function:1/ (10 s^2)**
- `Gm=tf([1],[0.1 1])` **=>Transfer function:1/(0.1 s + 1)**

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MATLAB: Block chart conversions

- `Gc=parallel(Gc1,Gc2)` **=>Transfer function:(31 s + 30)**
- `Gff=series(Gc,G)` **=>Transfer function:**
  - **(31 s + 30)/(10 s^2)**
- `Gtot=feedback(Gff,Gm)` **=>Transfer function:**

```
  3.1 s^2 + 34 s + 30
------------------------
s^3 + 10 s^2 + 31 s + 30
```

- Commands parallel and series can be replaced with + and * (parallel blocks are summed together and sequentially multiplied)
  - `Gc=Gc1+Gc2`
  - `Gff=Gc*G`
  - `Gtot=feedback(Gff,Gm)`
- The entire block chart conversion can also be done in one line:
  - `Gtot=feedback(G*(Gc1+Gc2),Gm)`

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MATLAB: Block chart conversions

- The total block diagram can also be calculated symbolically using the symbolic function in Matlab
- First we define *s* = tf('*s*')

And then

- `Gc1=31*s`
- `Gc2=30`
- `G=1/(10*s^2)`
- `Gm=1/(0.1*s+1)`
- `Gtot=(Gc1+Gc2)*G/(1+(Gc1+Gc2)*G*Gm)`
- `Gtot=minreal(Gtot)` *% Minreal reduces common terms*

```
  3.1 s^2 + 34 s + 30
-----------------------
s^3 + 10 s^2 + 31 s + 30
```

```
        (31 s + 30) (s + 10)
1/10 ------------------------
      (s + 5) (s + 3) (s + 2)
```

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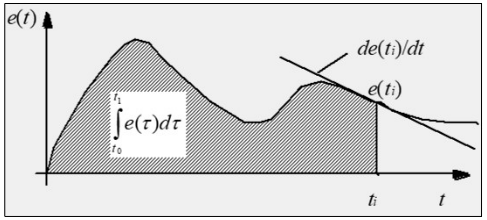
PID Controller

- The PID control is the most common regulator in the industry

$$\begin{cases} u(t) = K_P\left(e(t) + \dfrac{1}{T_I}\displaystyle\int_0^t e(\tau)d\tau + T_D\dfrac{de(t)}{dt}\right) \\ u(t) = K_P e(t) + K_I\displaystyle\int_0^t e(\tau)d\tau + K_D\dfrac{de(t)}{dt} \end{cases} \qquad \begin{cases} K_I = \dfrac{K_P}{T_I} \\ K_D = K_P T_D \end{cases}$$



PID Controller

- The PID controller input is $e(t)$ (deviation between desired and measured value $y_{ref}(t)$ - $y_{mit}(t)$) and the output is the control signal $u(t)$.
- The control provided by the Controller is the sum of three different functions, which are influenced by tuning parameters K_P, K_I and K_D

$$u(t) = u_P(t) + u_I(t) + u_D(t) = K_P e(t) + K_I\int_0^t e(\tau)d\tau + K_D\frac{de(t)}{dt}$$

  - The proportional term (P - proportional) is the static measure of the difference between desired and current value. Whenever you change the difference, the term u_P also changes in poroportion.
  - The integral term integrates the error. The u_I is in constant state of flux until error has disappeared. The integral term eliminates persistent deviation, but may increase system vibrations.
  - The derivative term follows the rate of change of the differential. Whenever the difference is changing, then u_D responds to resist the change. A derivative term stabilizes the system but is sensitive to delays and unwanted noise.

PID Controller-Integral term

$$u_I(t) = K_I\int_0^t e(\tau)d\tau$$

- The ability of the integral term to eliminate the permanent error can be illustrated by presenting it in a derivative form.

$$\frac{du_I(t)}{dt} = K_I e(t)$$

- Notice that the control u(s) continues to change until the error e(t) goes to zero.

Transfer function for PID controller

$$U(s) = G_{PID}(s)E(s), \quad G_{PID}(s) = K_P + K_I\frac{1}{s} + K_D s = K_P\left(1 + \frac{1}{T_I s} + T_D s\right)$$

- The PID control provides all basic modifications by multiplying non-preferred terms by zero. In the example of a Rocket control, the controller had a PD control. The same principle can also be used to form PI^2D control.
- P-control, K_P is to confirm the controller $G_P(s) = K_P$
- PI-control, K_P gives control proportional to error, K_I is the integral contstant and T_I is the integration time constant.

$$G_{PI}(s) = K_P + K_I\frac{1}{s} = K_P\left(1 + \frac{1}{T_I s}\right)$$

- The PD control, K_P gives control proportional to error, K_D is the derivative gain and T_D is the derivation time constant.

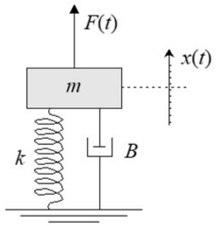
$$G_{PD}(s) = K_P + K_D s = K_P(1 + T_D s)$$

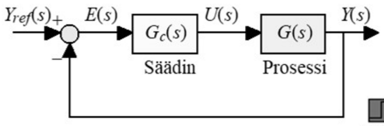
Example: Mechanical system

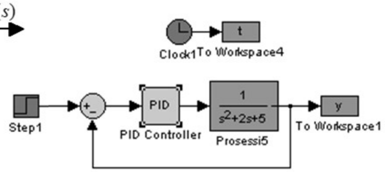
- Simulating the position of the mechanical system mass when controlled with various modifications of the PID controller. Assume the measurement and actuator to be ideal.

$$m\ddot{x}(t) + B\dot{x}(t) + kx(t) = F(t) \quad (k = 5, B = 2, m = 1)$$

$$\Rightarrow \quad G(s) = \frac{1}{s^2 + 2s + 5}$$







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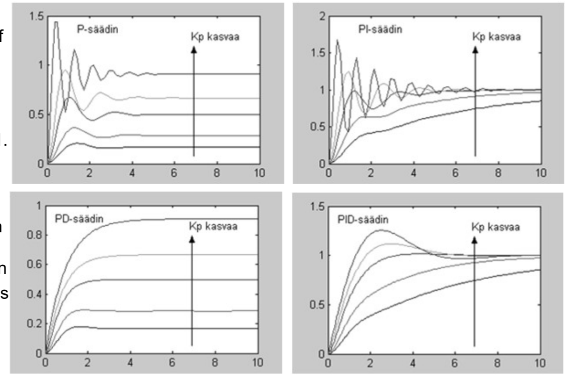
Example 2: Mechanical system

- The figures below show the step control results for P, PI, PD and PID
- The Simulations used proportional control's constant of the values 1, 2, 5, 10, and 50. The Integration and the derivatives times were simulated as 1.
- The P and PD controls leave a permanent deviation, the I-term eliminates the permanent deviation
- I-The term increases vibrations while the D-term stabilizes and eliminates vibrations



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Stability tests (Routh-Hurwitz)

- If the roots of the system are known (the denominator polynomial zero points), then stability is easy to observe.
  - Roots can be determined from numerical polynomial by iterative calculation routines (by Matlab commands such as **eig**, **roots** and **pole**).
    - E.g. Polynomial $s^3 + 2s^2 + 4s + 10$

```
roots([1 2 4 10])
ans =       -2.2236
             0.1118 + 2.1177i
             0.1118 - 2.1177i
```

  - If one of the polynomial coefficients is zero or negative, then the polynomial has at least one root on the imaginary axis or the right half plane.
  - If the polynomial contains symbolic parameters and you want to determine at which parameter values the system is stable, then the root solution numerically will no longer be successful. You can then use the Routh's chart.
  - The method is given in the following without proof (which would need a reasonable study of polynomial algebra).

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Routh's Chart

- Consider polynomial $a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$ to generate the Routh's chart::

s^n	a_0	a_2	a_4	a_6	a_8	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	a_9	$\cdots$
s^{n-2}	b_0	b_2	b_4	b_6	$\cdots$	
s^{n-3}	b_1	b_3	b_5	b_7	$\cdots$	
s^{n-4}	c_0	c_2	c_4	$\cdots$		
s^{n-5}	c_1	c_3	c_5	$\cdots$		
$\vdots$	$\vdots$					
s^1	z_0					
s^0	z_1					

$$b_0 = \frac{-1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}, \quad b_2 = \frac{-1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}, \quad b_4 = \frac{-1}{a_1}\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix}, \quad \cdots$$

$$b_1 = \frac{-1}{b_0}\begin{vmatrix} a_1 & a_3 \\ b_0 & b_2 \end{vmatrix}, \quad b_3 = \frac{-1}{b_0}\begin{vmatrix} a_1 & a_5 \\ b_0 & b_4 \end{vmatrix}, \quad b_5 = \frac{-1}{b_0}\begin{vmatrix} a_1 & a_7 \\ b_0 & b_6 \end{vmatrix}, \quad \cdots$$

$$c_0 = \frac{-1}{b_1}\begin{vmatrix} b_0 & b_2 \\ b_1 & b_3 \end{vmatrix}, \quad c_2 = \frac{-1}{b_1}\begin{vmatrix} b_0 & b_4 \\ b_1 & b_5 \end{vmatrix}, \quad c_4 = \frac{-1}{b_1}\begin{vmatrix} b_0 & b_6 \\ b_1 & b_7 \end{vmatrix}, \quad \cdots$$

$$\vdots$$

$$z_1 = a_n$$

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Routh's Chart

- The number of sign changes in the first column of the Routh's chart is also the number of roots at the right side of the complex plane.
- If the typical polynomial of the system is placed in the Routh diagram, the system is stable if there is no sign change in the first column.
- If there is a zero in the first column of the chart, it is replaced by the small positive number e in the diagram and forming diagram is continued. The final chart can be used to calculate the sign changes by examining the limits of the terms that depend on e ⟶ 0.
- If the chart consists of a whole row of zeros, then the original polynomial is divisible by another polynomial formed by the coefficients above the zero line.

Examples: Routh's chart

- Polynomials:
-

$$s^3+2s^2+4s+10$$

s^3	1	4
s^2	2	10
s^1	-1	
s^0	10	

Two sign changes 2→-1 and -1→10
So two roots on the right half-plane (RHP).

$$s^4+4s^3+6s^2+4s+2$$

s^4	1	6	2
s^3	4	4	
s^2	5	2	
s^1	$12/5$		
s^0	2		

No sign changes in first column
So no roots on the right half-plane

Examples: Routh's chart

- Polynomial: s^3+s^2+2s+2

s^3	1	2
s^2	1	2
s^1	0	0
s^0		

A zero row is obtained, resulting in a higher line polynomial s^2+2 with which the original polynomial is divisible.

Another way: Take the derivative of the auxiliary polynomial and continue.

$$\frac{d}{ds}(s^2+2)=2s$$

s^3	1	2
s^2	1	2
s^1	2	0
s^0	2	

No character changes, so no roots on the right side of the plane

Examples: Routh's chart

- Polynomial: $s^4+3s^3+4s^2+12s+12$

s^4	1	4	12
s^3	3	12	
s^2	$0\to\varepsilon$	12	
s^1	$(12\varepsilon-36)/\varepsilon$		
s^0	12		

The first column becomes zero, replace it with a low positive number ε and continue

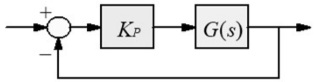
$$\lim_{\varepsilon\to 0}\left\{\frac{12\varepsilon-36}{\varepsilon}\right\}=-\infty$$

s^4	1	4	12
s^3	3	12	
s^2	0	12	
s^1	$-\infty$		
s^0	12		

Two sign changes 0→-∞ ja -∞→12
⇒ Two roots on the right side of the plane

Examples: Routh's chart

- System with the transfer function $G(s) = \dfrac{1}{s^3 + 4s^2 + s - 6}$ is controlled with the P controller.
- For what values of K_P is the system stable?



$$G_{TOT}(s) = \frac{K_P G(s)}{1 + K_P G(s)} = \frac{\dfrac{K_P}{s^3 + 4s^2 + s - 6}}{1 + \dfrac{K_P}{s^3 + 4s^2 + s - 6}} = \frac{K_P}{s^3 + 4s^2 + s + (K_P - 6)}$$

s^3	1	1
s^2	4	$K_P - 6$
s^1	$\dfrac{10 - K_P}{4}$	
s^0	$K_P - 6$	

Stable if, $\dfrac{10 - K_P}{4} \geq 0$ ja $K_P - 6 \geq 0$

$$\Rightarrow 6 \leq K_P \leq 10$$

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State space poles and zeros

- Earlier the conversion between transfer function and state space representation was derived.

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

- The inverse of the matrix is calculated by dividing the adjoint matrix with the determinant.

$$\mathbf{G}(s) = \frac{\mathbf{C}\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B}}{\det(s\mathbf{I} - \mathbf{A})} + \mathbf{D} = \frac{\mathbf{C}\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + \mathbf{D}\det(s\mathbf{I} - \mathbf{A})}{\det(s\mathbf{I} - \mathbf{A})}$$

Characteristic polynomial: $\det(s\mathbf{I} - \mathbf{A}) = 0$

System poles equation: $\det(s\mathbf{I} - \mathbf{A}) = 0$

System zeros equation: $\mathbf{C}\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + \mathbf{D}\det(s\mathbf{I} - \mathbf{A}) = 0$

- Note that the equations apply to multivariable (MIMO) systems as well.

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Example

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) = \begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}\mathbf{u}(t) \\ y(t) = \mathbf{C}\mathbf{x}(t) = \begin{bmatrix} 1 & 1 \end{bmatrix}\mathbf{x}(t) \end{cases}$$

- Determine system poles and zeros.

Characteristic equation:

$$\det(s\mathbf{I} - \mathbf{A}) = \det\begin{bmatrix} s+1 & -2 \\ 0 & s-3 \end{bmatrix} = (s+1)(s-3) = 0$$

$$\mathbf{C}\,\mathrm{adj}(s\mathbf{I} - \mathbf{A})\mathbf{B} + \mathbf{D}\det(s\mathbf{I} - \mathbf{A}) = \begin{bmatrix} 1 & 1 \end{bmatrix}\mathrm{adj}\begin{bmatrix} s+1 & -2 \\ 0 & s-3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} s-3 & 2 \\ 0 & s+1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2(s+3) & s-3 \end{bmatrix}$$

$$\Rightarrow \quad \mathbf{G}(s) = \begin{bmatrix} \dfrac{2(s+3)}{(s+1)(s-3)} & \dfrac{s-3}{(s+1)(s-3)} \end{bmatrix} = \begin{bmatrix} \dfrac{2(s+3)}{(s+1)(s-3)} & \dfrac{1}{s+1} \end{bmatrix}$$

- The transfer function has two poles (-1 ja 3) and one Zero (-3)

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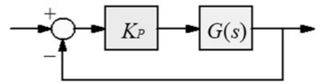
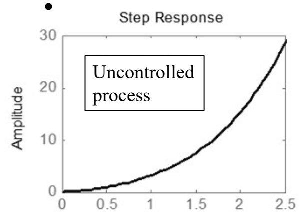
Example: Behavior of a given system

- Examine the behavior of the prescribed system with the various tuning controls (P-control).

$$G(s) = \frac{s+2}{s^2 - s} = \frac{s+2}{s(s-1)}$$



- The uncontrolled process is unstable
-



Regulated system

$$G_{TOT}(s) = \frac{K_P G(s)}{1 + K_P G(s)} = \frac{K_P(s+2)}{s^2 - s + K_P(s+2)}$$

$$= \frac{K_P(s+2)}{s^2 + (K_P - 1)s + 2K_P}$$

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Example: Behavior of the given system

- Characteristic equation of the given system $s^2 + (K_P - 1)s + 2K_P = 0$
- System poles (Quadratic equation solution):

$$s_{p,1,2} = -\frac{K_P - 1}{2} \pm \frac{\sqrt{K_P^2 - 10K_P + 1}}{2}$$

- The system is not oscillating when the poles are
- $$K_P^2 - 10K_P + 1 \geq 0 \;\Rightarrow\; \left(K_P - 5 + 2\sqrt{6}\right)\left(K_P - 5 - 2\sqrt{6}\right) \geq 0$$

$$\Rightarrow\; K_P \leq 5 - 2\sqrt{6} \approx 0.1010 \quad \text{or} \quad K_P \geq 5 + 2\sqrt{6} \approx 9.8990$$

- The system is stable when the poles are on the left half plane

$$K_P \geq 0 \quad \text{ja} \quad K_P - 1 \geq 0 \;\Rightarrow\; K_P \geq 1$$

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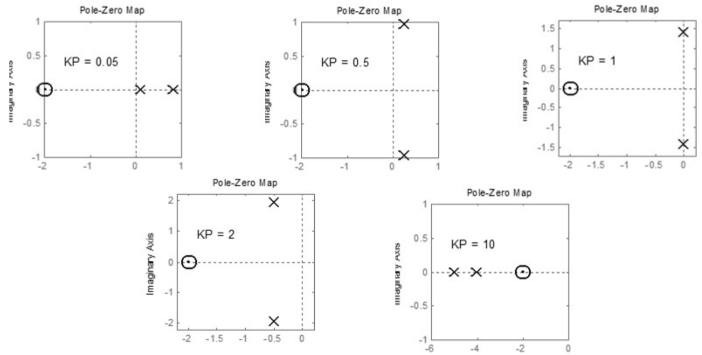
Example: Behavior of the given system

- Determine the poles and plot the pole zero maps of the given system with different K_P values



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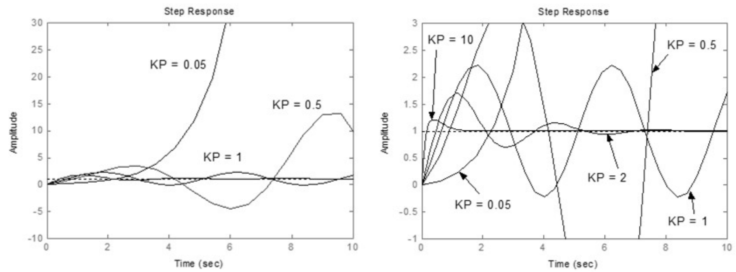
Example: Behavior of the given system

- The corresponding step responses are:



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Example: Behavior of the given system

- So, a controlled system can be

$K_P \leq 5 - 2\sqrt{6}$,	Response is non-oscillatory and unstable
$5 - 2\sqrt{6} \leq K_P < 1$,	Response has oscillations with growing amplitude and unstable
$K_P = 1$,	Response has harmonic oscillations
$1 < K_P < 5 + 2\sqrt{6}$,	Response has oscillations but is stable
$K_P = 5 + 2\sqrt{6}$,	Response is critically stable
$K_P > 5 + 2\sqrt{6}$,	Response has no oscillations and is stable

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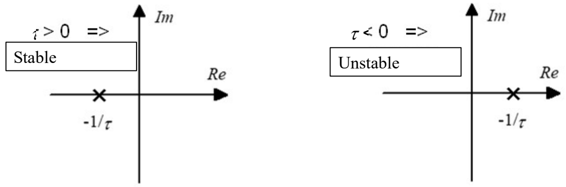
Common transfer function templates: 1st order system

- 1. First order dynamics
  - Differential equation and transfer function:
    - *K is the gain*
    - τ is the system time constant

$$\tau\dot{y}(t)+y(t)=Ku(t) \qquad G(s)=\frac{K}{\tau s+1}$$

Impulse response $y(t)=\frac{K}{\tau}e^{-\frac{t}{\tau}}$

Step response $y(t)=K\left(1-e^{-\frac{t}{\tau}}\right)$



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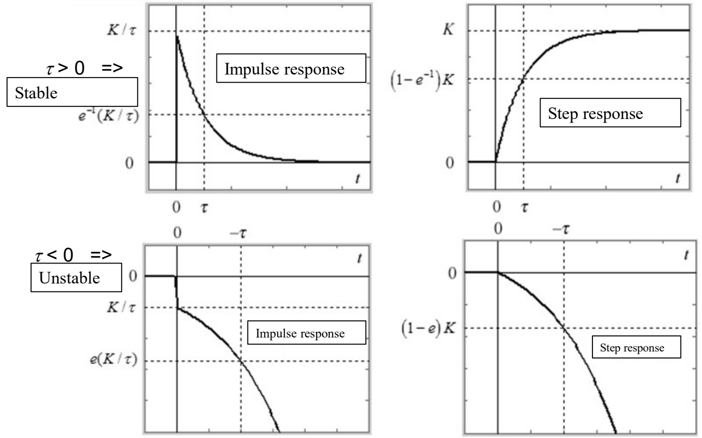
Common transfer function templates: 1st order system



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Common transfer function templates: 2nd order system

- 2nd order oscillation dynamics (complex poles)
  - Differential equation and transfer function:
    - *K is the system gain*
    - ω_n is system natural angular frequency ($\omega_n > 0$)
    - ζ is the damping ratio of the system (-1 > ζ > 1)

$$\ddot{y}(t)+2\zeta\omega_n\dot{y}(t)+\omega_n^2y(t)=K\omega_n^2u(t)$$

$$G(s)=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Impulse response : $y(t)=\frac{K\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left(\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right)$

Step response : $y(t)=K\left(1-e^{-\zeta\omega_n t}\left(\cos\left(\omega_n\sqrt{1-\zeta^2}t\right)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)\right)\right)$

$$=K\left(1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\left(\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)+\cos^{-1}(\zeta)\right)\right)$$

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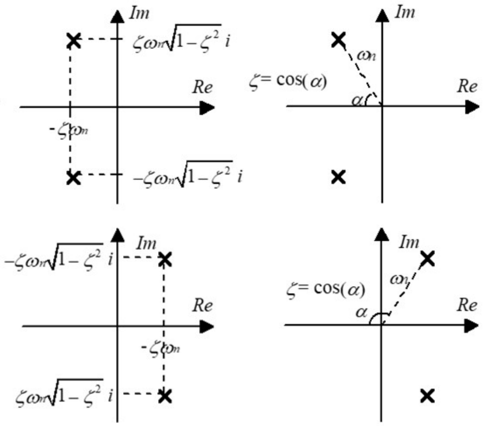
Common transfer function templates: 2nd order system

poles:

$$\begin{cases} s_{P,1}=-\omega_n\zeta+\omega_n\sqrt{1-\zeta^2}\,i \\ s_{P,2}=-\omega_n\zeta-\omega_n\sqrt{1-\zeta^2}\,i \end{cases}$$

0 > ζ > 1 =>
Stable

-1 > ζ > 0 =>
Unstable



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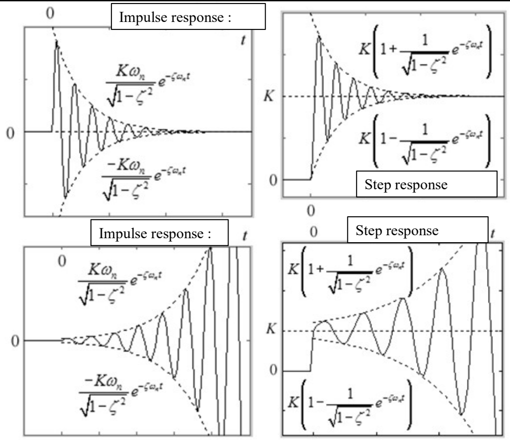
Common transfer function templates: 2nd order system

$0 > \zeta > 1$ =>
Stable

$-1 > \zeta > 0$ =>
Unstable



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Common transfer function templates: 2nd order system

- The oscillation dynamics of 2nd order system (real poles)
  - Differential equation and transfer function:
    - K is system control gain
    - τ_1 and τ_2 are system time constants ($\tau_1 \neq \tau_2$)

$$\tau_1\tau_2\ddot{y}(t)+(\tau_1+\tau_2)\dot{y}(t)+y(t)=Ku(t)$$

$$G(s)=\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}$$

Impulse response : $y(t)=\frac{K}{\tau_2-\tau_1}\left(e^{-\frac{t}{\tau_2}}-e^{-\frac{t}{\tau_1}}\right)$

Step response : $y(t)=K\left(1-\frac{1}{\tau_2-\tau_1}\left(\tau_2 e^{-\frac{t}{\tau_2}}-\tau_1 e^{-\frac{t}{\tau_1}}\right)\right)$

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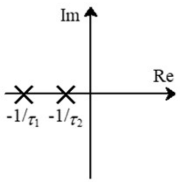
Common transfer function templates: 2nd order system

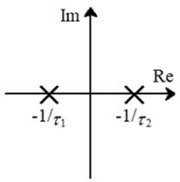
poles:

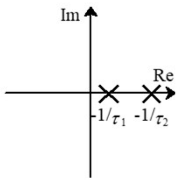
$$\begin{cases} s_{P,1}=-\frac{1}{\tau_1} \\ s_{P,2}=-\frac{1}{\tau_2} \end{cases}$$

$\tau_1, \tau_2 > 0$
=> stable



$\tau_1 < 0$ tai $\tau_2 < 0$
=> Unstable





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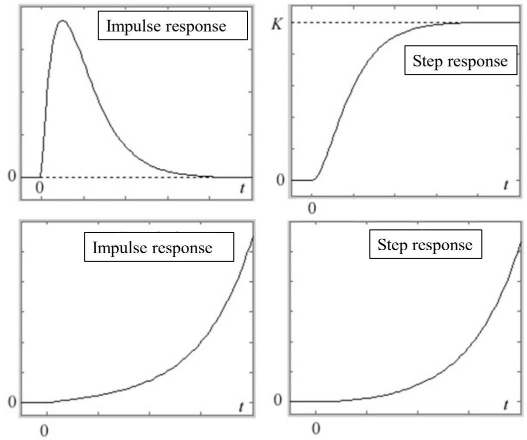
Common transfer function templates: 2nd order system

$\tau_1, \tau_2 > 0$
=> stable

$\tau_1 < 0$ or $\tau_2 < 0$
=> unstable



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Higher-order Models

- Higher-order models can be formed from simple 1st and 2nd order models
- It is known that the response is a weighted sum of all elements (poles and zeros near the vertical axis dominate the behavior of the system)
  - Examining the oscillating system of the 3rd order:

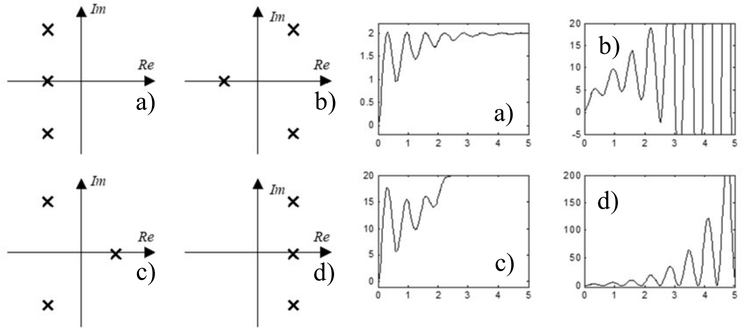
$$G(s)=\frac{K\omega_n^2}{(\tau s+1)(s^2+2\xi\omega_n s+\omega_n^2)}$$

  - The behavior of this system is a weighted sum of second-order behavior and first-order behavior:

$$G(s)=\frac{K\omega_n^2}{(\tau s+1)(s^2+2\xi\omega_n s+\omega_n^2)}=\frac{A}{\tau s+1}+\frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

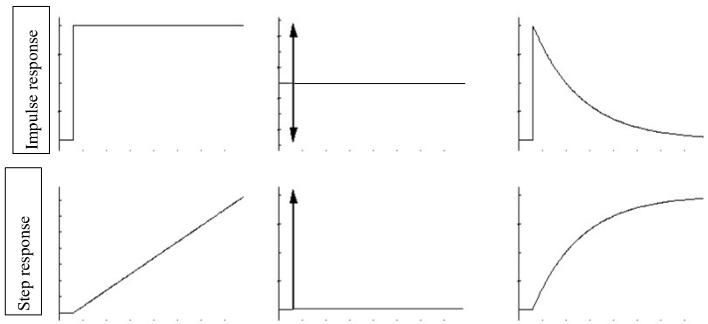
Higher-order Models

- Below are system pole zero patterns and step responses with certain parameter values



Simple system responses

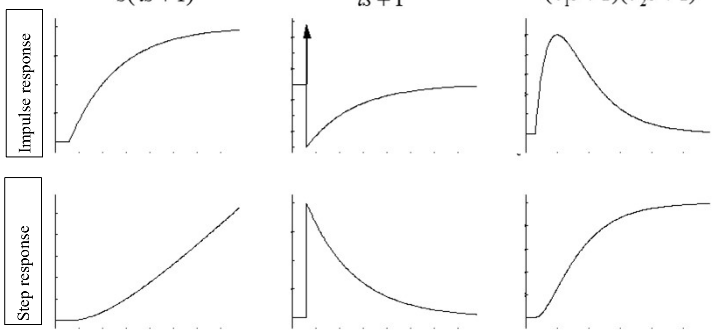
$\frac{K}{s}$ Ks $\frac{K}{\tau s+1}$



$\tau>0,\ K>0,\ 0<\zeta<1,\ \omega_n\neq 0$

Simple system responses

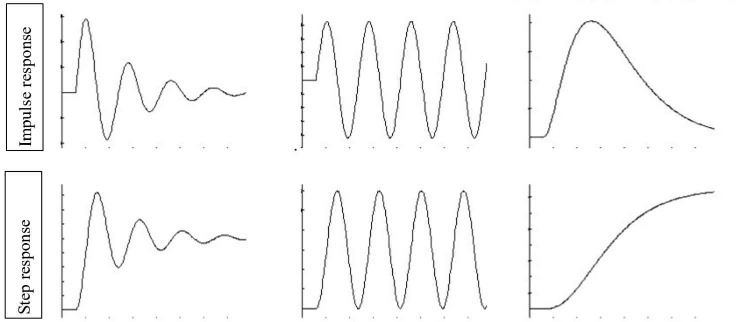
$\frac{K}{s(\tau s+1)}$ $\frac{Ks}{\tau s+1}$ $\frac{K}{(\tau_1 s+1)(\tau_2 s+1)}$



$\tau>0,\ K>0,\ 0<\zeta<1,\ \omega_n\neq 0$

Simple system responses

$$\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad \frac{K\omega_n^2}{s^2+\omega_n^2} \qquad \frac{K}{(\tau_1 s+1)(\tau_2 s+1)(\tau_3 s+1)}$$



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$\tau > 0,\ K > 0,\ 0 < \zeta < 1,\ \omega_n \neq 0$

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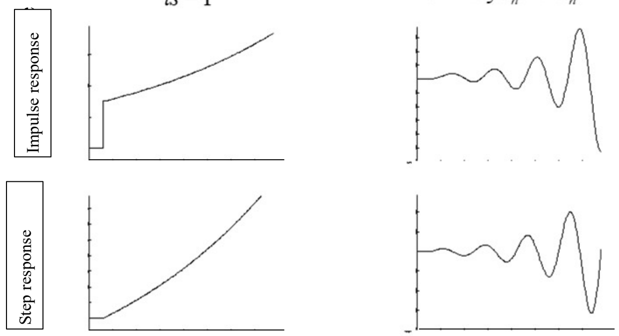
Simple system responses

$$\frac{K}{\tau s-1} \qquad \frac{K\omega_n^2}{s^2-2\zeta\omega_n s+\omega_n^2}$$



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$\tau > 0,\ K > 0,\ 0 < \zeta < 1,\ \omega_n \neq 0$

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