1. In a block diagram, there are overlapping inter-block relations, so one must either rely on a mathematical solution or modify the block diagram to determine the total transfer function, so that overlapping structures are removed.

   To solve a total transfer function using intermediate variables:

   Select intermediate variables $\varepsilon_1(s)$, $\varepsilon_2(s)$ and $\varepsilon_3(s)$ as shown in the diagram. (Other choices are possible, for example: instead of writing $\varepsilon_3(s)$ explicitly, we could write it in the form $1/s \cdot \varepsilon_2$.)
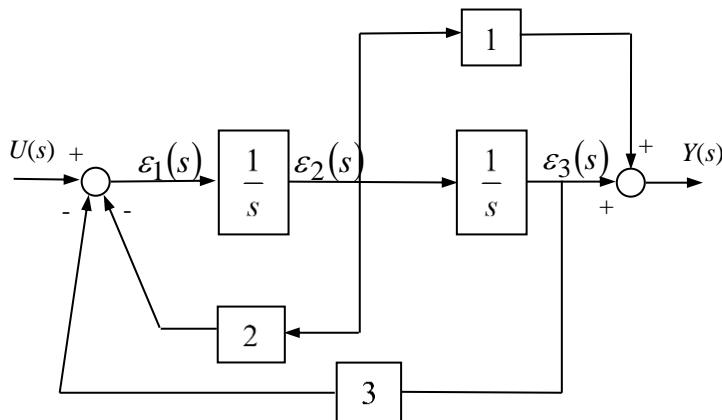
   Generating equations for each of the intermediate quantities $\varepsilon_i(s)$ and output $Y(s)$:

   $$\Rightarrow \begin{cases} \varepsilon_1(s) = U(s) - 2\varepsilon_2(s) - 3\varepsilon_3(s) \\ \varepsilon_2(s) = \dfrac{1}{s}\varepsilon_1(s) \\ \varepsilon_3(s) = \dfrac{1}{s}\varepsilon_2(s) = \dfrac{1}{s^2}\varepsilon_1(s) \\ Y(s) = \varepsilon_2(s) + \varepsilon_3(s) \end{cases}$$

   Eliminating $\varepsilon_2(s)$ and $\varepsilon_3(s)$ by writing them in terms of $\varepsilon_i(s)$:

   $$\Rightarrow \begin{cases} \varepsilon_1(s) = U(s) - \dfrac{2}{s}\varepsilon_1(s) - \dfrac{3}{s^2}\varepsilon_1(s) \\ Y(s) = \dfrac{1}{s}\varepsilon_1(s) + \dfrac{1}{s^2}\varepsilon_1(s) = \left(\dfrac{1}{s} + \dfrac{1}{s^2}\right)\varepsilon_1(s) \end{cases}$$

   $$\Rightarrow \begin{cases} \left(1 + \dfrac{2}{s} + \dfrac{3}{s^2}\right)\varepsilon_1(s) = U(s) \\ Y(s) = \left(\dfrac{1}{s} + \dfrac{1}{s^2}\right)\varepsilon_1(s) \end{cases}$$

$$\Rightarrow \begin{cases} \varepsilon_1(s) = \dfrac{1}{\left(1 + \dfrac{2}{s} + \dfrac{3}{s^2}\right)} U(s) \\[4mm] Y(s) = \left(\dfrac{1}{s} + \dfrac{1}{s^2}\right)\varepsilon_1(s) \end{cases}$$

Eliminating $\varepsilon_1(s)$

$$Y(s) = \frac{\dfrac{1}{s} + \dfrac{1}{s^2}}{1 + \dfrac{2}{s} + \dfrac{3}{s^2}} U(s) = \frac{s+1}{s^2 + 2s + 3} U(s)$$

Thus, the overall transfer function is:

$$G_{TOT}(s) = \frac{Y(s)}{U(s)} = \frac{s+1}{s^2 + 2s + 3}$$

2. System:
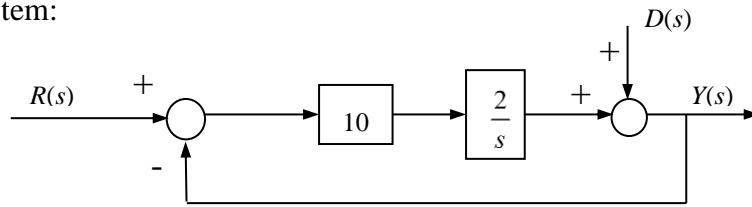
From the block diagram, we can write:

$$E(s) = R(s) - Y(s) = R(s) - \left[\frac{20}{s}E(s) + D(s)\right] \Rightarrow \left(1 + \frac{20}{s}\right)E(s) = R(s) - D(s)$$

$$\Rightarrow E(s) = \frac{R(s) - D(s)}{1 + 20/s}$$

$$Y(s) = R(s) - E(s) = \frac{(1 + 20/s)R(s) - R(s) + D(s)}{1 + 20/s} = \frac{(20/s)R(s) + D(s)}{1 + 20/s} = \frac{sD(s) + 20R(s)}{s + 20} \quad .$$

Laplace transformation gives the values of $D(s)$ and $R(s)$ as (from the table):

$$D(s) = 5\frac{s}{s^2 + 1}, \quad R(s) = 5\frac{1}{s} \quad .$$

Thus, the output in Laplace form is:

$$Y(s) = \frac{\dfrac{5s^2}{s^2 + 1} + \dfrac{100}{s}}{s + 20} = \frac{5s^3 + 100s^2 + 100}{s(s^2 + 1)(s + 20)} \quad .$$

Taking partial fractions:

$$\frac{5s^3 + 100\,s^2 + 100}{s(s^2+1)(s+20)} \equiv \frac{A}{s} + \frac{Bs+C}{s^2+1} + \frac{D}{s+20} = \frac{A(s^2+1)(s+20) + (Bs+C)s(s+20) + Ds(s^2+1)}{s(s^2+1)(s+20)}$$

$$\Rightarrow \begin{cases} A+B+D = 5 \\ 20A + 20B + C = 100 \\ A + 20C + D = 0 \\ 20A = 100 \end{cases} \Rightarrow \begin{cases} A = 5 \\ B = 5/401 \\ C = -100/401 \\ D = -5/401 \end{cases}$$

This gives:

$$Y(s) = \frac{5}{s} + \frac{5}{401}\cdot\frac{s}{s^2+1} - \frac{100}{401}\cdot\frac{1}{s^2+1} - \frac{5}{401}\cdot\frac{1}{s+20},$$

Taking inverse Laplace transform, we get in time domain:

$$y(t) = 5 + \frac{5}{401}\cos(t) - \frac{100}{401}\sin(t) - \frac{5}{401}e^{-20t}$$

Perfoming the same conversions in Matlab:

```
>> G = tf({[1 0] 20},{[1 20] [1 20]})
 Transfer function from input 1 to output:
  s
------
s + 20
 Transfer function from input 2 to output:
  20
------
s + 20
```

Simulated response to lsim command: (matlab: Help lsim)

```
>> t = 0:0.05:20; R = 5*ones(1,401); D = 5*cos(t);
>> lsim(G,[D;R],t);
```


Linear Simulation Results
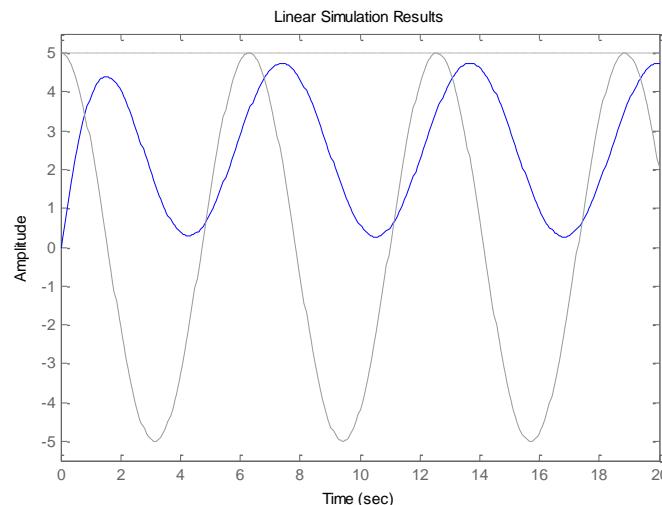
3. State Space form:

$$\begin{cases} \dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\mathbf{x}(t) \end{cases}$$

Corresponding transfer function: $\dfrac{Y(s)}{U(s)} = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}$

$$\Rightarrow s\mathbf{I}\text{-}\mathbf{A}=\begin{bmatrix} s-1 & 0 & -2 \\ -1 & s-1 & 0 \\ 0 & -2 & s-1 \end{bmatrix} \Rightarrow \det(s\mathbf{I}-\mathbf{A}) = (s-1)^3 - 2\cdot(-1)\cdot(-2) = s^3 - 3s^2 + 3s - 5$$

$$\mathrm{adj}(s\mathbf{I}-\mathbf{A}) = \begin{bmatrix} \begin{vmatrix} s-1 & 0 \\ -2 & s-1 \end{vmatrix} & -\begin{vmatrix} -1 & 0 \\ 0 & s-1 \end{vmatrix} & \begin{vmatrix} -1 & s-1 \\ 0 & -2 \end{vmatrix} \\ -\begin{vmatrix} 0 & -2 \\ -2 & s-1 \end{vmatrix} & \begin{vmatrix} s-1 & -2 \\ 0 & s-1 \end{vmatrix} & -\begin{vmatrix} s-1 & 0 \\ 0 & -2 \end{vmatrix} \\ \begin{vmatrix} 0 & -2 \\ s-1 & 0 \end{vmatrix} & -\begin{vmatrix} s-1 & -2 \\ -1 & 0 \end{vmatrix} & \begin{vmatrix} s-1 & 0 \\ -1 & s-1 \end{vmatrix} \end{bmatrix}^{\mathrm{T}} =$$

$$=\begin{bmatrix} s^2-2s+1 & s-1 & 2 \\ 4 & s^2-2s+1 & 2s-2 \\ 2s-2 & 2 & s^2-2s+1 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} s^2-2s+1 & 4 & 2s-2 \\ s-1 & s^2-2s+1 & 2 \\ 2 & 2s-2 & s^2-2s+1 \end{bmatrix}$$

Bearing in mind that $(s\mathbf{I}-\mathbf{A})^{-1} = \dfrac{\mathrm{adj}(s\mathbf{I}\text{-}\mathbf{A})}{\det(s\mathbf{I}-\mathbf{A})}$, the transfer function is obtained as:

$$\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} = \frac{1}{s^3-3s^2+3s-5}\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} s^2-2s+1 & 4 & 2s-2 \\ s-1 & s^2-2s+1 & 2 \\ 2 & 2s-2 & s^2-2s+1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} =$$

$$= \frac{s^2-2s+1}{s^3-3s^2+3s-5}$$

Same calculation in Matlab:
```
>> A = [1 0 2; 1 1 0; 0 2 1]; B = [1 0 0]'; C = [1 0 0]; D = 0;
>> [num,den] = ss2tf(A,B,C,D)

num =

        0    1.0000   -2.0000    1.0000


den =

   1.0000   -3.0000    3.0000   -5.0000
```

Same result, another method:
```
>> A = [1 0 2; 1 1 0; 0 2 1]; B = [1 0 0]'; C = [1 0 0]; D = 0;
>> G1=ss(A,B,C,D); G2=tf(G1); [num,den]=tfdata(G2,'v');
```

4.   **a.** Polynomial: $s^4 + 6s^3 + 13s^2 + 12s + 4$

Corresponding Routh's chart:

s^4	1	13	4
s^3	6	12	0
s^2	$\dfrac{6 \cdot 13 - 1 \cdot 12}{6} = 11$	$\dfrac{6 \cdot 4 - 1 \cdot 0}{6} = 4$	
s^1	$\dfrac{11 \cdot 12 - 6 \cdot 4}{11} = \dfrac{108}{11}$	0	
s^0	$\dfrac{108/11 \cdot 4 - 11 \cdot 0}{108/11} = 4$		

The first column is: [1 6 11 108/11 4] . There are no sign changes in the first column, so all poles in the polynomial are on the left side.

b. Polynomial: $2s^5 + s^4 + 3s^2 + s + 2 = 2s^5 + s^4 + 0s^3 + 3s^2 + s + 2$

Corresponding Routh's chart:

s^5	2	0	1
s^4	1	3	2
s^3	$\dfrac{1 \cdot 0 - 2 \cdot 3}{1} = -6$	$\dfrac{1 \cdot 1 - 2 \cdot 2}{1} = -3$	
s^2	$\dfrac{-6 \cdot 3 - 1 \cdot (-3)}{-6} = \dfrac{5}{2}$	$\dfrac{-6 \cdot 2 - 1 \cdot 0}{-6} = 2$	
s^1	$\dfrac{5/2 \cdot (-3) - (-6) \cdot 2}{5/2} = \dfrac{9}{5}$	0	
s^0	$\dfrac{9/5 \cdot 2 - 5/2 \cdot 0}{9/5} = 2$		

The first column is: [2 1 -6 5/2 9/5 2] . There are two sign changes: $1 \rightarrow$ -6 and $-6 \rightarrow 5/2$, So the polynomial has two poles on the right side of the plane.

c. Polynomial: $s^4 + 2s^3 + 4s^2 + 8s + 10$

Corresponding Routh's chart:

s^4	1	4	10
s^3	2	8	0
s^2	$\dfrac{2 \cdot 4 - 1 \cdot 8}{2} = \varepsilon$	$\dfrac{2 \cdot 10 - 1 \cdot 0}{2} = 10$	
s^1	$\dfrac{\varepsilon \cdot 8 - 2 \cdot 10}{\varepsilon} = c_0$	0	
s^0	$\dfrac{c_0 \cdot 10 - \varepsilon \cdot 0}{c_0} = 10$		

In the term $c_0 = \dfrac{\varepsilon \cdot 8 - 2 \cdot 10}{\varepsilon}$ The limit value when ε approaches 0:

$$\lim_{\varepsilon \to 0}\{c_0\} = \lim_{\varepsilon \to 0}\left\{\frac{8\varepsilon - 20}{\varepsilon}\right\} = 8 - 20\lim_{\varepsilon \to 0}\left\{\frac{1}{\varepsilon}\right\} = -\infty \ .$$

Thus, the first column becomes: [1 2 0 -∞ 10] . Now the first column has two sign changes: $0 \to -\infty$ and $-\infty \to 10$, So the polynomial has two poles on the right side of the plane.

Matlab solves for the roots quickly using `roots`-command, for example, take the above polynomial :

```
>> roots([1 2 4 8 10])

ans =

   0.4199 + 1.8588i
   0.4199 - 1.8588i
  -1.4199 + 0.8588i
  -1.4199 - 0.8588i
```

5.   A system with an unstable process:
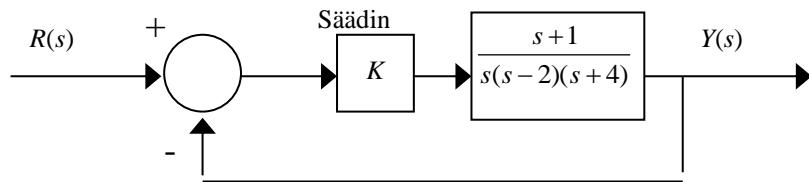
Generating closed loop system transfer function:

$$\frac{Y(s)}{R(s)} = \frac{K\dfrac{s+1}{s(s-2)(s+4)}}{1+K\dfrac{s+1}{s(s-2)(s+4)}} = \frac{K(s+1)}{s(s-2)(s+4)+K(s+1)} = \frac{K(s+1)}{s^3 + 2s^2 + (K-8)s + K}$$

Let's look at the pole positions by making the Routh chart for polynomial $s^3 + 2s^2 + (K-8)s + K$:

s^3	1	K-8
s^2	2	K
s^1	$\dfrac{2\cdot(K-8)-1\cdot K}{2} = \dfrac{K-16}{2}$	0
s^0	$\dfrac{(K-16)/2\cdot K - 2\cdot 0}{(K-16)/2} = K$	

Stability condition: $\dfrac{K-16}{2} > 0$ and $K > 0$

$\Rightarrow$ The system is asymptotically stable when $K > 16$. An unstable system can therefore be well controlled by feedback and proportional (P) control.