ELEC-C8201: Control Theory and Automation

Exercise 5

The problems marked with an asterisk (⋆) are not discussed during the exercise session. The solutions are given in MyCourses and these problems belong to the course material.

1. A feedback control system with a proportional gain 4 and a plant with transfer function

$$G(s) = \frac{s^2+1}{s(s+a)}$$

is shown in Figure 1.

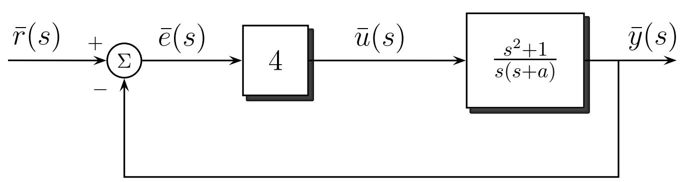

Figure 1: Feedback control system.

Sketch the root locus for $0 \le a < \infty$.

Solution. We first restructure our system to be in the way we know how to handle:

$$1 + k_p G(s) = 0 \Rightarrow 1 + 4\frac{s^2+1}{s(s+a)} = 0$$

$$\Rightarrow s(s+a) + 4(s^2+1) = 0$$

$$\Rightarrow as + 5s^2 + 4 = 0$$

$$\Rightarrow 1 + a\frac{s}{5s^2+4} = 0$$

Next, we find the break-in point:

$$\sum_i \frac{1}{\sigma - z_i} = \sum_j \frac{1}{\sigma - p_j} \Rightarrow \frac{1}{\sigma} = \frac{1}{\sigma + j\beta} + \frac{1}{\sigma - j\beta}$$

$$\Rightarrow \frac{1}{\sigma} = \frac{2\sigma}{\sigma^2 + 4/5}$$

$$\Rightarrow \sigma^2 + 4/5 = 2\sigma^2 \Rightarrow \sigma^2 = 4/5 \Rightarrow \sigma = \pm 2/\sqrt{5}$$

Since $a > 0$ and by the 2nd step of the root locus procedure, the break-in happens at $\sigma = -2/\sqrt{5} \approx 0.8944$.

Hence, the root locus plot is given by

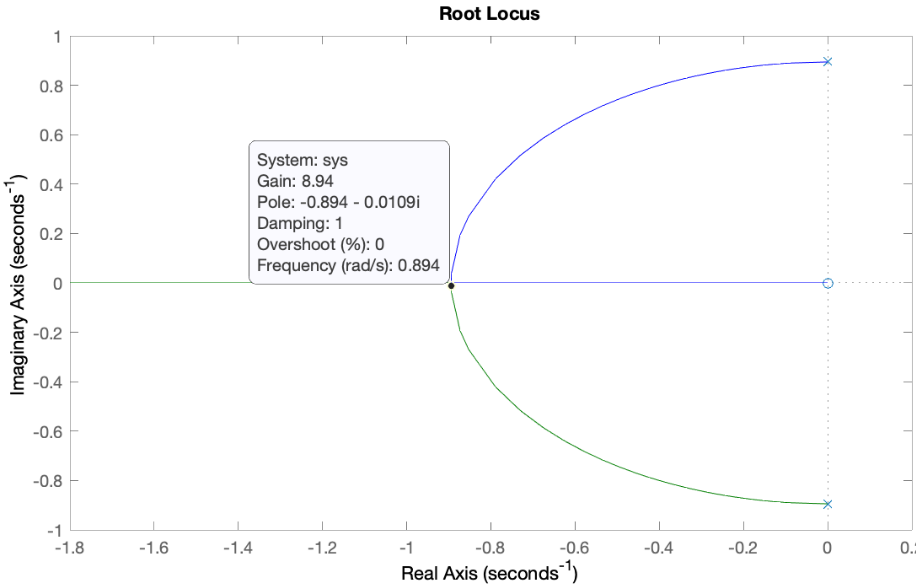

Figure 2: Root locus for changing a.

MATLAB Code:

```
sys = tf([1 0],[5 0 4]); % Defines the transfer function
rlocus(sys); % Produces the root locus plot
```

2. A feedback control system with a plant transfer function

$$G(s) = \frac{1}{s(s-1)}$$

is shown in Figure 3.

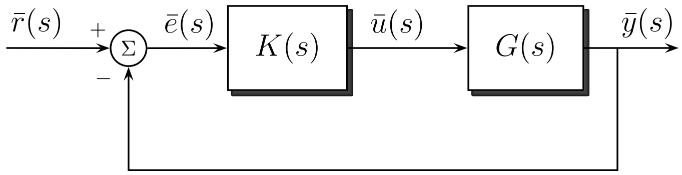

Figure 3: Feedback control system.

a) When $K(s) = k_p$, show that the system is always unstable by sketching the root locus.

b) When

$$K(s) = \frac{k_p(s+2)}{s+20},$$

sketch the root locus and determine the range of k_p for which the system is stable.

Solution.

a)

$$1 + k_p G(s) = 0 \Rightarrow 1 + k_p \frac{1}{s(s-1)} = 0$$

$$\Rightarrow s^2 - s + k_p = 0$$

$$\Rightarrow s_{1,2} = \frac{1 \pm \sqrt{1-4k_p}}{2}$$

From the roots one can see that for $k_p = 0$ one of the roots is at zero and as k_p increases $\sqrt{1-4k_p} < 1$ for all k_p such that $1 - 4k_p > 0$. Once $k_p > 1/4$, we will have to poles whose real part is always at $-1/2$.

One can see this from the root locus (Figure 4) as well:

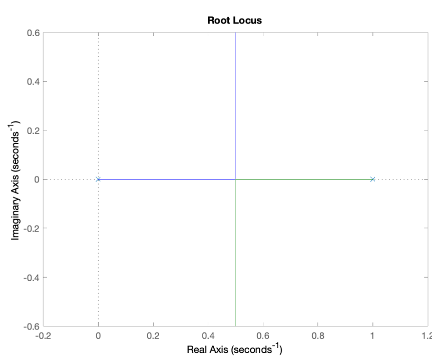

Figure 4: Root locus.

b)

$$1 + K(s)G(s) = 0 \Rightarrow 1 + \frac{k_p(s+2)}{s+20}\frac{1}{s(s-1)} = 0$$

$$\Rightarrow 1 + k_p\frac{(s+2)}{s(s-1)(s+20)} = 0$$

Then, one can find the intersection with the $j\omega$-axis and the asymptote centroid and the asymptotes and construct the root locus plot.

The root locus plot (Figure 5) is plotted here with MATLAB:

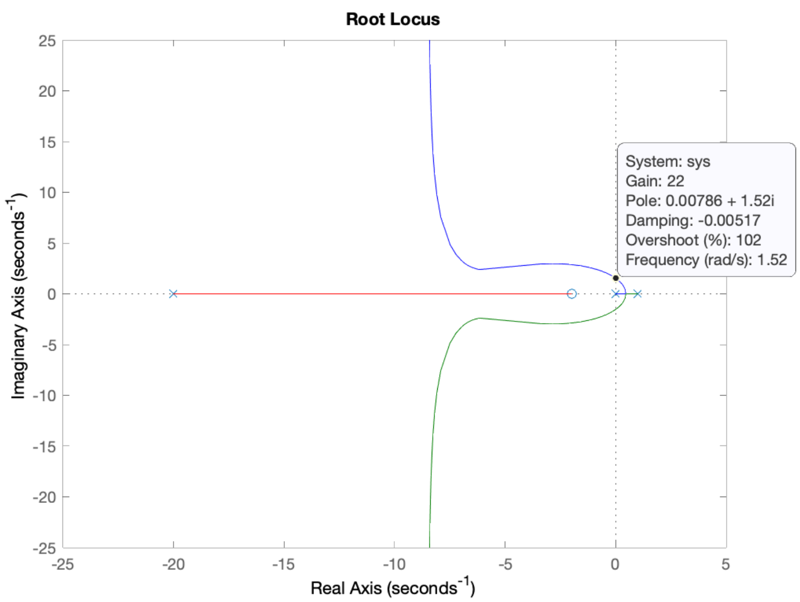

Figure 5: Root locus.

MATLAB Code:

```
s = tf('s')
sys = (s+2)/(s*(s-1)*(s+20))  % Defines the transfer function
rlocus(sys); % Produces the root locus plot
```

3. A feedback control system with a proportional gain k_p and a plant with transfer function

$$G(s) = \frac{s+10}{s(s+5)}$$

is shown in Figure 6.

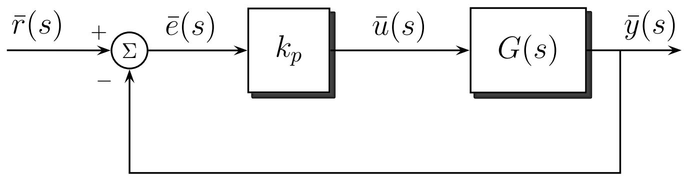

Figure 6: Feedback control system.

a) Determine the break-in and break-away points of the root locus and sketch the root locus for $k_p > 0$.

b) Determine the gain k_p when the two characteristic roots have a damping factor ζ of $1/\sqrt{2}$.

c) Calculate the roots.

Solution.

a) We find the break-in and break-away points as follows:

$$\sum_i \frac{1}{\sigma - z_i} = \sum_j \frac{1}{\sigma - p_j} \Rightarrow \frac{1}{\sigma+10} = \frac{1}{\sigma} + \frac{1}{\sigma+5}$$

$$\Rightarrow \frac{1}{\sigma+10} = \frac{2\sigma+5}{\sigma(\sigma+5)}$$

$$\Rightarrow \sigma^2 + 5\sigma = 2\sigma^2 + 25\sigma + 50$$

$$\Rightarrow (s+10)^2 - 50 = 0 \Rightarrow s_{1,2} = -10 \pm \sqrt{50}$$

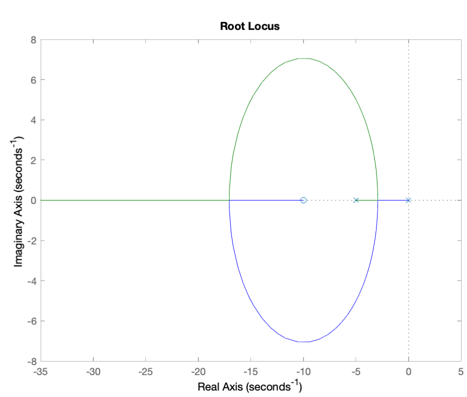

Figure 7: Root locus.

b)

$$1 + K(s)G(s) = 0 \Rightarrow 1 + k_p \frac{(s+10)}{s(s+5)} = 0$$

$$\Rightarrow s^2 + (5 + k_p)s + 10k_p = s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Therefore,

$$\begin{cases} 2\zeta\omega_n = 5 + k_p \\ \omega_n^2 = 10k_p \end{cases} \Rightarrow \begin{cases} \omega_n = \sqrt{10k_p} \\ \zeta = \frac{5+k_p}{2\sqrt{10k_p}} \end{cases}$$

For $\zeta = 1/\sqrt{2} = \sqrt{2}/2$,

$$\frac{5 + k_p}{2\sqrt{10k_p}} = \frac{\sqrt{2}}{2} \Rightarrow 20k_p = (5 + k_p)^2$$

$$\Rightarrow k_p^2 - 10k_p + 5^2 = 0 \Rightarrow (k_p - 5)^2 = 0 \Rightarrow k_p = 5$$

c) For $k_p = 5$ we substitute to the characteristic equation and

$$s^2 + 10s + 50 = 0 \Rightarrow (s + 5)^2 + 5^2 = 0 \Rightarrow s_{1,2} = -5 \pm 5j$$

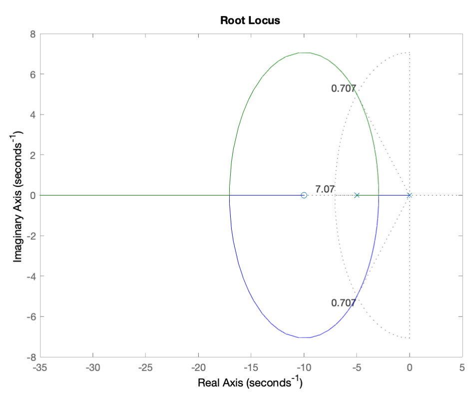

Figure 8: Root locus.

MATLAB Code:

```
s = tf('s')
sys = (s+10)/(s*(s+5))
rlocus(sys)
sgrid(1/sqrt(2),sqrt(50)) % plots a grid of damping factor zeta and
    natural frequency wn, respectively
```

4. A feedback control system with a proportional gain k_p and a plant with transfer function

$$G(s) = \frac{(s+2)^2}{s(s^2+1)(s+8)}$$

is shown in Figure 6.

a) First, sketch the root locus for $0 \leq k_p < \infty$ to indicate the significant features of the locus. Second, use MATLAB to plot the root locus and compare it with your sketch.

b) For what value of k_p do purely imaginary roots exist?

c) Determine the range of the gain k_p for which the system is stable.

d) Would the use of the dominant roots approximation for an estimate of the settling time be justified in this case for a large magnitude of gain ($k_p > 50$)?

Solution.

a) Root locus is given by

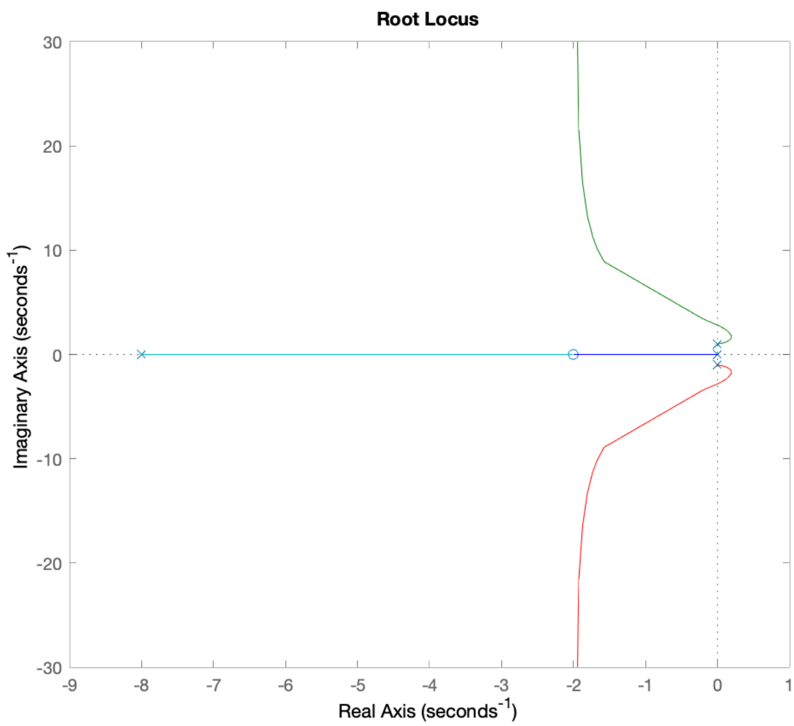

Figure 9: Root locus.

Remark. Note that it is difficult while sketching to predict how the poles from the imaginary axis will move. However, if one tries to determine the $j\omega$ crossings, it will become apparent.

b) On the imaginary axis, we know that the roots satisfy

$$1 + k_p G(j\omega) = 0 \Rightarrow 1 + k_p \frac{(j\omega + 2)^2}{j\omega((j\omega)^2 + 1)(j\omega + 8)} = 0$$

which after algebraic manipulation and by splitting the real and imaginary parts, we get

$$\begin{cases} -\omega^2(1-\omega^2) + k_p(4-\omega^2) = 0 \\ 8\omega(1-\omega^2) + 4k_p\omega = 0 \end{cases}$$

Solving for k_p, we get $k_p = 14$

c) From b) and the root locus diagram, it is obvious that for $k_p > 14$ the system will be stable.

d) When $K > 50$, the real part of the complex roots is approximately equal to the real part of the two real roots and therefore the complex roots are not dominant roots.

⋆5. A magnetically levitated (MAGLEV) high-speed train "flies" on an air gap above its rail system (with up to 310mph!), as shown in Figure 10.

Figure 10: A MAGLEV train in China (Photo: Ren Long/China Features Photos).

The feedback control system is illustrated in Figure 11.

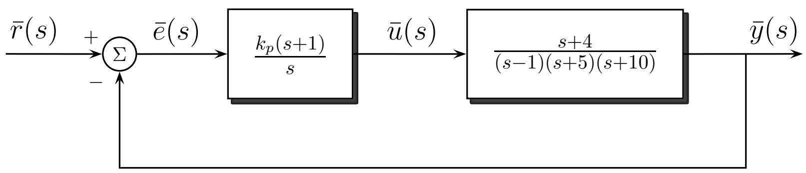

Figure 11: Feedback control system.

a) Sketch the root locus plot.

b) Select k_p so that all of the complex roots have a damping factor ζ greater than 0.6. Plot (in MATLAB) the actual response for the selected k_p.

c) Select k_p so that the response for a unit step input is reasonably damped and the settling time is less than 5 seconds. Plot (in MATLAB) the actual response for the selected k_p.

Solution.

a) First, we should notice that we can write it in the following form

$$1 + K(s)G(s) = 0 \Rightarrow 1 + k_p \frac{(s+1)(s+4)}{s(s-1)(s+5)(s+10)} = 0$$

Then, we can sketch the root locus, or, plot it in MATLAB

```
s = tf('s')
sys = (s+1)*(s+4)/(s*(s-1)*(s+5)*(s+10))
rlocus(sys)
```

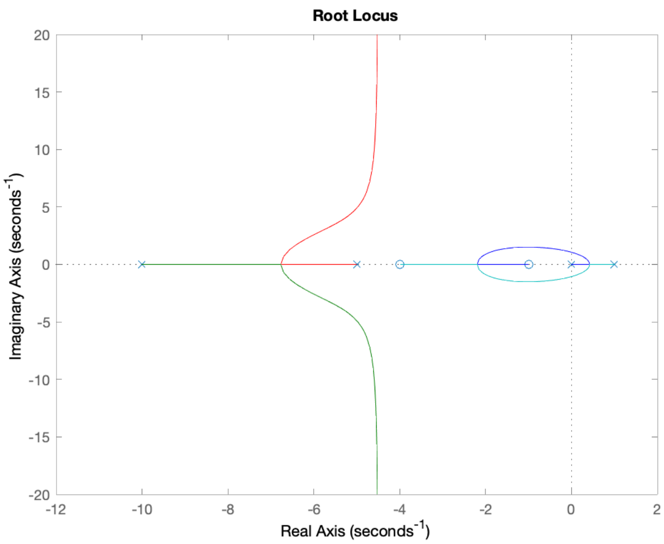

Figure 12: Root locus.

b) To have a damping factor ζ greater than 0.6, then we mast have a gain that is greater that ~ 44 and smaller than ~ 75; see the root locus in below

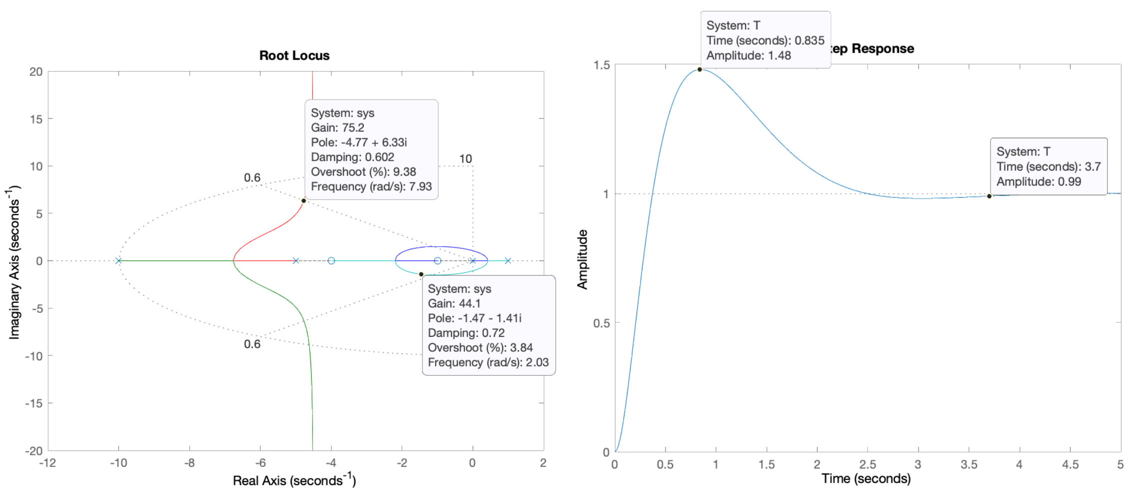

Figure 13: Root locus and step response ($k_p = 44$).

```
s = tf('s')
sys = 44*(s+1)*(s+4)/(s*(s-1)*(s+5)*(s+10))
T = feedback(ss(sys),1);
step(T)
```

c) To get the minimum settling time we have to be as far from the imaginary axis as possible. Hence, based on the root locus diagram we can choose $k_p \approx 71$ and obtain a better performance:

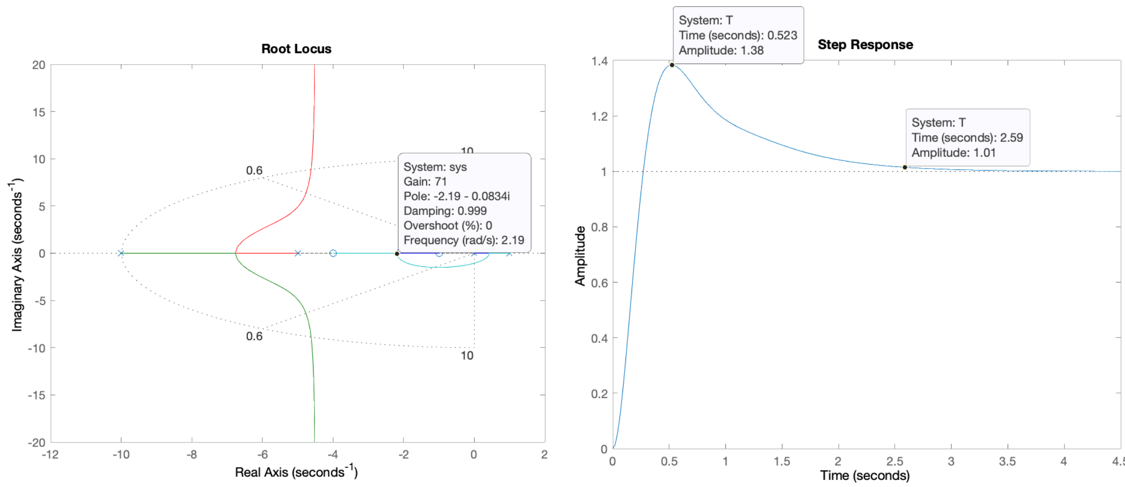

Figure 14: Root locus and step response ($k_p = 44$).

```
s = tf('s')
sys = 71*(s+1)*(s+4)/(s*(s-1)*(s+5)*(s+10))
T = feedback(ss(sys),1);
step(T)
```