Differential and Integral Calculus 3, MS-A0311

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Office Hour: Fridays 11:00-12:00 on Zoom

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Two ways to get a grade on the course

I $\underbrace{\text{Exercise sessions, Hand-ins}}_{50\%}$ & $\underbrace{\text{course exam}}_{50\%}$

Exercise sessions:

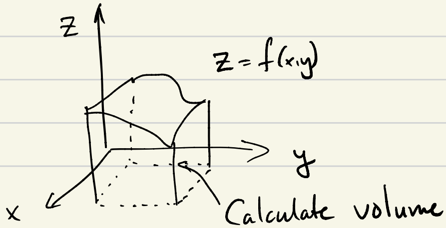
	H01	H02	
	Wednesday,	Tuesday	Home work (Deadline Mondays Midnights)
	Friday,	Friday	Demo ex.

Hand-ins: Deadline Wednesdays, Midnight

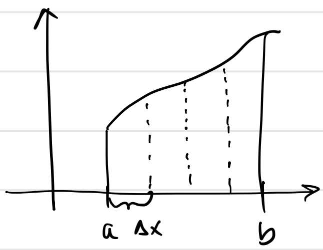
II Exam 100%

Multiple integrals

First double integrals

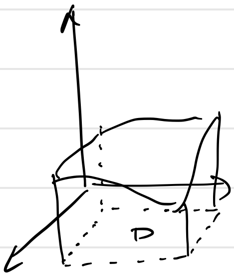


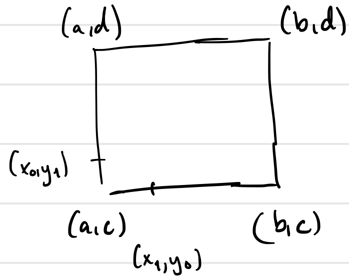
In the one-dimensional case this is an area calculation



$$\text{Area} = \int_a^b f(x)\,dx = $$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

In the two-dimensional case the idea is the same.





$$\text{Volume} = \iint_D f(x,y)\,dA = \lim_{n\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} f(x_i,y_j)\,\Delta x\,\Delta y$$

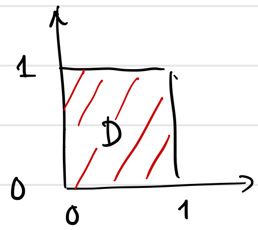
$$\Delta x = \frac{b-a}{n} \; ; \; \Delta y = \frac{d-c}{n}$$

For double (and multiple) integrals the fundamental theorem of calculus does not immediately generalize. However one can calculate them using iterated integrals and here primitive functions comes into play again.

Ex $f(x,y) = xy^2$

Calculate $\iint_D f(x,y)\,dA$ where

$D = \{ (x,y) \in \mathbb{R}^2 ;\ 0 \le x \le 1,\ 0 \le y \le 1 \}$



$$\iint_D xy^2\,dA = \int_0^1 \left(\int_0^1 xy^2\,dy \right) dx =$$

$$= \int_0^1 \left[\frac{xy^3}{3} \right]_{y=0}^{y=1} dx = \int_0^1 \frac{x}{3}\,dx$$

$$= \left[\frac{x^2}{6} \right]_0^1 = \frac{1}{6}.$$

Triple integrals (and multiple integrals in general) works in the same way. The construction works in the same way and you can calculate them as iterated integrals.

Ex $\iiint_S xye^z\,dV \qquad S = \{ 0 \le x \le 2,\ 0 \le y \le 1,\ -1 \le z \le 1 \}$

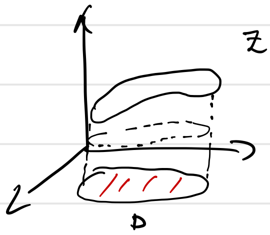
$$\iiint_S xye^z\,dV = \int_{-1}^1 dz \int_0^1 dy \int_0^2 xye^z\,dx =$$

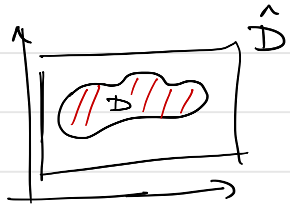
$$= \int_{-1}^1 dz \int_0^1 dy \left[\frac{x^2 ye^z}{2} \right]_0^2 =$$

$$= \int_{-1}^1 dz \int_0^1 2ye^z\,dy = \int_{-1}^1 \left[y^2 e^z \right]_0^1 dz =$$

$$= \int_{-1}^1 e^z\,dz = \left[e^z \right]_{-1}^1 = e - e^{-1}$$

How to integrate over more general domains



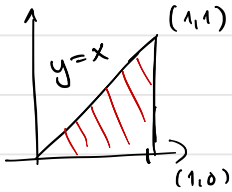


$$\hat{f}(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in \hat{D} - D. \end{cases}$$

$$\iint_D f(x,y)\, dA := \iint_{\hat{D}} \hat{f}(x,y)\, dA.$$

In practice you do as in the following example.

Ex Let D be as in the figure



$$D = \{ 0 \le x \le 1,\ 0 \le y \le x \}$$

$$\iint_D xy\, dA = \int_0^1 dx \int_0^x xy\, dy = \int_0^1 \left[\frac{xy^2}{2} \right]_{y=0}^{y=x} dx =$$

$$= \int_0^1 \frac{x^3}{2}\, dx = \left[\frac{x^4}{8} \right]_0^1 = \frac{1}{8}$$

Also, $D = \{ 0 \le y \le 1,\ y \le x \le 1 \}$ so

$$\iint_D xy\, dA = \int_0^1 dy \int_y^1 xy\, dx = \int_0^1 \left[\frac{x^2 y}{2}\right]_{x=y}^{x=1} dy =$$

$$= \int_0^1 \frac{y}{2} - \frac{y^3}{2}\, dy = \left[\frac{y^2}{4} - \frac{y^4}{8}\right]_0^1 = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}.$$

In theory the order of integration doesn't matter. However in practice it can make a big difference.

Ex $\iint_D e^{x^2}\, dA = ?$ D as before

$$\iint_D e^{x^2} dA = \int_0^1 dx \int_0^x e^{x^2} dy = \int_0^1 x e^{x^2} dx =$$

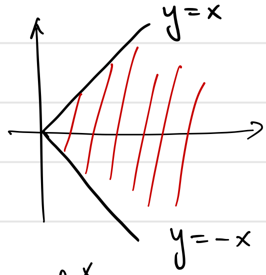
$$= \left\lceil \begin{matrix} t = x^2 \\ dt = 2x\,dx \end{matrix} \right\rfloor = \frac{1}{2}\int_0^1 e^t\, dt = \left[\frac{1}{2} e^t\right]_0^1 = \frac{e-1}{2}$$

However $\int_0^1 \int_y^1 e^{x^2} dx dy$ is not possible to calculate using elementary functions

Improper integrals

The domain of integration and/or integrand can be unbounded. We only consider positive integrands in what follows $(f \geq 0)$ since they can be handled using iterated integrals.

Ex Let R be defined as



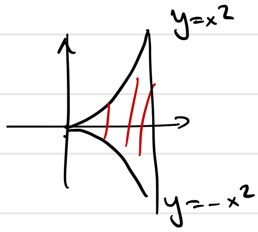
$I = \iint_R e^{-x^2}\, dA$ is an improper integral

$$I = \int_0^{\infty} dx \int_{-x}^{x} e^{-x^2} dy = \int_0^{\infty} 2xe^{-x^2} dx = \lim_{N\to\infty} \int_0^N 2xe^{-x^2} dx$$

$$= \begin{bmatrix} t = x^2 & t_N = N^2 \\ dt = 2x\,dx & t_0 = 0 \end{bmatrix} = \lim_{N\to\infty} \int_0^{N^2} e^{-t} dt =$$

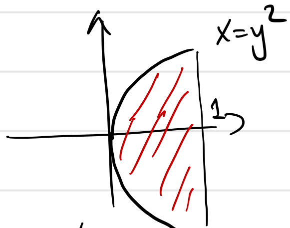
$$= \lim_{N\to\infty} \left[-e^{-t}\right]_0^{N^2} = \lim_{N\to\infty} 1 - e^{-N^2} = 1$$

Ex Let R be



$$\iint_R \frac{1}{x^2}\, dA = \int_0^1 dx \int_{-x^2}^{x^2} \frac{1}{x^2} dy = \int_0^1 2\, dx = 2$$

Now change R to be



$$\iint_R \frac{1}{x^2}\, dA = \int_0^1 dx \int_{-\sqrt{x}}^{\sqrt{x}} \frac{1}{x^2} dy = \int_0^1 2\sqrt{x}\, \frac{1}{x^2} dx =$$

$$= \int_0^1 2x^{-3/2} dx = \lim_{\varepsilon\to 0^+} \int_{\varepsilon}^1 2x^{-3/2} dx =$$

$$= \lim_{\varepsilon\to 0^+} \left[\frac{2x^{-1/2}}{-1/2}\right]_{\varepsilon}^1 = \lim_{\varepsilon\to 0^+} \frac{4}{\sqrt{\varepsilon}} - 4 = \infty$$

Integrability depends on both $f(x,y)$ and R.