Circuit elements

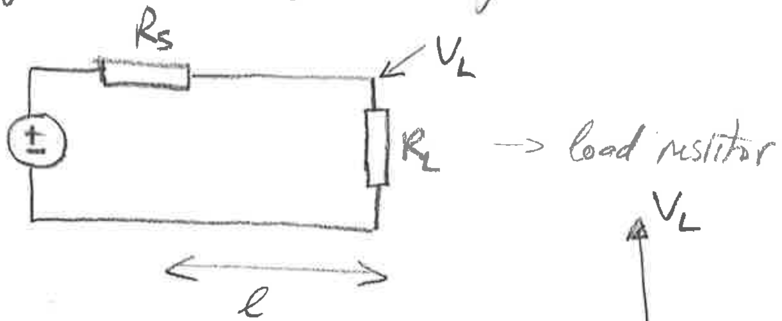
- The electromagnetic spectrum as seen by a quantum engineer:

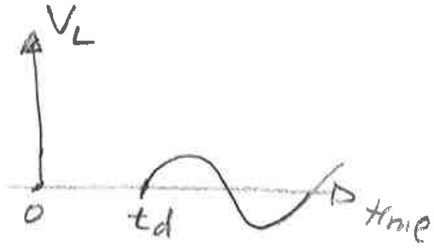
Low frequency (audio/rf)	Microwaves	THz	Optical	X-rays, γ-rays
Kirchhoff's laws; lumped circuit elements	Distributed-circuit models; electromagnetic simulations; superconducting qubits	Nobody knows...	Geometrical optics. Quantum optics: photons, cavities, atoms, etc.	None of my business!

frequency →

1 GHz ($\lambda = 30\,\mathrm{cm}$) 300 GHz ($\lambda = 1\,\mathrm{mm}$)

- Why lumped-circuit models don't work at high frequencies?
  The speed of light c is large but finite.





$t_d = \frac{\ell}{c}$ = delay time

becomes non-negligible if

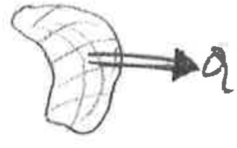
$\lambda \sim \ell \sim$ cm

frequency $\sim$ GHz

Some basic concepts - electrical circuits

- electric current

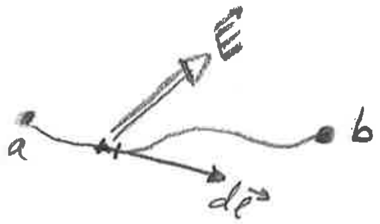
$$I = \frac{dQ}{dt} \qquad Q = \int_0^t I\,dt$$



by convention, the direction of the current is the direction of motion of positive charges.

- work done by an electric field

$$W_{ba} = q \int_a^b \vec{E} \cdot d\vec{l}$$



- voltage

$$V = V_{ba} = V_b - V_a = -\frac{dW_{ba}}{dq} = -\int_a^b \vec{E}\, d\vec{l}$$

W_{ba} = work done to transport the charge q against the field, $W \equiv -W_{ba}$ = work done by the field

E.g. capacitor:

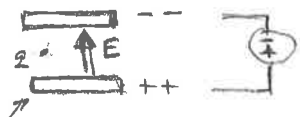


plate with higher electrical potential

- magnetic flux

Node flux: $\phi(t) = \int_{-\infty}^{t} d\tau\, V(\tau)$

so $V(t) = \frac{d\phi(t)}{dt}$

- power $\quad W(t) = \int_0^t P(\tau)\, d\tau$

$$P(t) = \frac{dW(t)}{dt} = \frac{dW(t)}{dq(t)} \cdot \frac{dq(t)}{dt} = V(t) \cdot I(t)$$

so $\quad W(t) = \int_0^t V(\tau) \cdot I(\tau)\, d\tau$

- phasors

Useful concept if:
- the circuit is linear
- all independent sources are sinusoidal
- only steady-state response is desired

$$X(t) = A\cos(\omega t + \phi) = Re(Ae^{i\phi}e^{i\omega t})$$

$X \equiv Ae^{i\phi} \equiv$ phasor = transformation of a sine waveform from time-domain into frequency domain.

Why it is useful? Simple rules:

variable	phasor
$X(t)$	$Ae^{i\phi}$
$dX(t)/dt$	$i\omega \cdot Ae^{i\phi}$
$\int dt X(t)$	$\frac{Ae^{i\phi}}{i\omega}$

check them based on the definition

- impedance and admittance

$$Z = \frac{V}{I}$$ with V and I phasors

$$Z = R + iX$$

Z = impedance
R = resistance
X = reactance

units: Ω (Ohm)

$$Y = \frac{1}{Z} = G + iB$$

Y = admittance
G = conductance
B = susceptance

units: S (Siemens)

- ac power and decibels

Suppose $V(t) = V_0 \cos(\omega t + \varphi_V) = \mathrm{Re}\left[V_0 e^{i\varphi_V} e^{i\omega t}\right]$ · phasor $V_0 e^{i\varphi_V} \equiv V$

$I(t) = I_0 \cos(\omega t + \varphi_I) = \mathrm{Re}\left[I_0 e^{i\varphi_I} e^{i\omega t}\right]$ · phasor $I_0 e^{i\varphi_I} \equiv I$

$$\Rightarrow P(t) = V(t) \cdot I(t) = \frac{1}{2} I_0 V_0 \cos(\varphi_V - \varphi_I) + \frac{1}{2} I_0 V_0 \cos(2\omega t + \varphi_V + \varphi_I)$$

instantaneous power

this term oscillates and cancels out when averaging

Average power:

$$\overline{P} = \frac{1}{T}\int_0^T P(t)\,dt = \frac{1}{2} I_0 V_0 \cos(\varphi_V - \varphi_I)$$

average power delivered to a load can be changed by changing the phases!

$$\overline{P} = \frac{1}{2}\mathrm{Re}\left[V \cdot I^*\right]$$

Root-mean square of a periodic signal

$I(t) = I \cos\omega t \longrightarrow I_{rms}^2 \equiv \frac{1}{T}\int_0^T I^2(t)\,dt = \frac{I_0^2}{2}$

use $\cos^2\omega t = \frac{1+\cos 2\omega t}{2}$

$I_{rms} = \frac{I_0}{\sqrt{2}}$

decibel:

$N(dB) = 10 \log_{10} \frac{P}{P_{reference}}$

power in decibels

P = power

$P_{reference}$ = a reference power, usually 1mW

If $P_{reference} = 1mW$, then $N(dBm) = 10 \log_{10} \frac{P}{1 mW}$

units are "dBm".

Since $P \sim V^2$, we

have $20 \log_{10} \frac{V}{V_{reference}}$ (in dBV) as another way to express this.

Examples:

30dB is an increase in power by 1000
20dB — — " — 100
10 dB — " — 10
3dB — " — — 2
0 dB — " — 1
-3 dB is a decrease in power by 2
-10dB — " — by 10

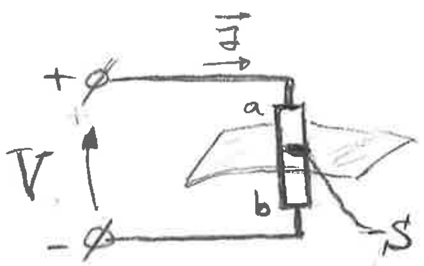
... etc.

Circuit elements

Note: an rf-circuit can be constructed from discrete (lumped) elements if the size of each component ≪ wavelength of the rf field

RESISTOR

Typically a film of conductive material evaporated on a chip.



remember that $\vec{E} = -\vec{\nabla} V$

$$V \equiv V_a - V_b = -\int_b^a \vec{E}\, d\vec{l} = \int_a^b \frac{\vec{J}}{\sigma} d\vec{l}$$

$\vec{J} = \sigma \cdot \vec{E}$ Ohm's law

σ = conductivity

If the frequency is not too high, then $\vec{J}$ = uniform over the cross-section S of the resistor

$$J = I/S$$

So $V = I \int_a^b \frac{1}{\sigma S} dl = I R$ $\qquad R = \frac{1}{\sigma S} \int_a^b dl$

$\rho = \frac{1}{\sigma}$ = resistivity

$\int_a^b dl = l$ = length of the device

$$\Rightarrow R = \rho \frac{l}{S}$$

and $\boxed{V = IR}$

$G = \frac{1}{R}$ = conductance

R = resistance

and $\boxed{Z(\omega) = R}$ – real and frequency independent

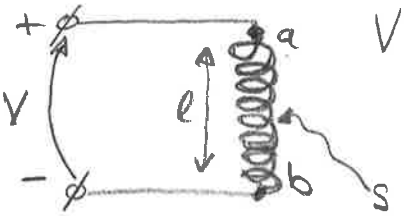
- Instantaneous dissipated power

$$P(t) = V(t) \cdot I(t) = R \cdot I^2(t) = G \cdot V^2(t)$$

- Average dissipated power (for harmonic excitations, $V(t) = V_0 \cos \omega t$)

$$\bar{P} = \frac{1}{2} R |I_0|^2 = \frac{1}{2} G |V_0|^2$$

INDUCTOR



$$V = V_{ab} = V_a - V_b = -\int_b^a \vec{E}\, d\vec{\ell} = \iint \frac{d\vec{B}(t)}{dt} d\vec{s} = L\frac{dI}{dt}$$

Maxwell-Faraday equation

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

+ Stokes theorem

Here we assumed $\vec{B}$ uniform over the surface of area S.

$B = \frac{\mu_0 \mu_r N I}{\ell}$ $\longrightarrow$ and there are N surfaces of area S

$\Longrightarrow$ $L = \frac{\mu_0 \mu_r N^2 S}{\ell}$

N = no. of turns of the solenoid

μ_0 = free-space magnetic permeability

μ_r = relative permeability

- Instantaneous energy stored

$$W_L(t) = \frac{1}{2} L I^2(t)$$

- Average energy stored in ac-harmonic fields

$$\overline{W_L} = \frac{1}{4} L I_0^2$$

So $\boxed{V = L\frac{dI}{dt}}$

therefore $Z_L(\omega) = iL\omega$

because $Z_L(\omega) = \frac{V(\omega)}{I(\omega)}$

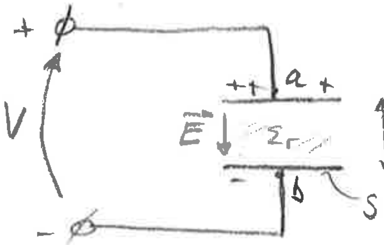
Note: The flux variable $\phi \equiv \iint \vec{B}\, d\vec{s}$ can be used to define a flux at a node with potential V,

$$\int_{-\infty}^{t} V(\tau)\, d\tau = \phi(t)$$

since $V(t) = \frac{d\phi(t)}{dt}$

We also have $I(t) = \frac{\phi(t)}{L}$.

CAPACITOR



$$V = -\int_b^a \vec{E}\, d\vec{\ell} = \frac{Q}{C}$$

C = capacitance

$$C = \frac{\varepsilon_0 \varepsilon_r S}{\ell}$$

Proof: $E = \frac{Q}{S\varepsilon_0\varepsilon_r} = \frac{V}{\ell} \Rightarrow V = \frac{Q}{\frac{\varepsilon_0 \varepsilon_r S}{\ell}}$

ε_0 = free-space electric permittivity $\quad$ ε_r = relative permittivity

Also $I(t) = \frac{dQ(t)}{dt} \Rightarrow \quad I(t) = C \cdot \frac{dV(t)}{dt}$

- Instantaneous energy stored

$$W_C(t) = \frac{1}{2} C V^2(t)$$

- Average energy stored for ac-harmonic fields

$$\overline{W_C} = \frac{1}{4} C V_0^2$$

So $\boxed{I = C \frac{dV}{dt}}$

therefore

$$Z_C(\omega) = \frac{1}{i\omega C}$$

again from $Z_C(\omega) = \frac{V(\omega)}{I(\omega)}$ and using the properties of phasors.

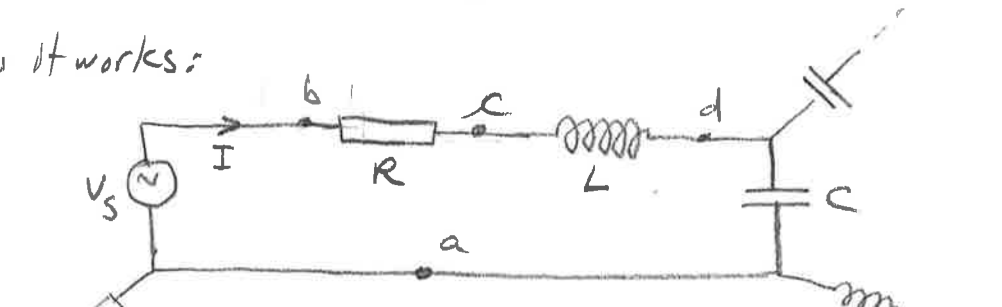
More complex networks of inductors, capacitors, resistors ...

= Kirchhoff's voltage and current laws =

- Kirchhoff's voltage law: FOR ANY CLOSED LOOP OF A CIRCUIT, THE ALGEBRAIC SUM OF VOLTAGES OF THE INDIVIDUAL BRANCHES IS ZERO

$$\sum V_K = 0$$

How it works:



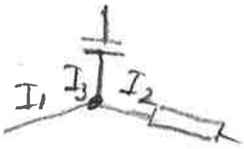
$$-\int_a^b \vec{E} d\vec{l} - \int_b^c \vec{E} d\vec{l} - \int_c^d \vec{E} dl - \int_d^a \vec{E} dE = 0$$

or $V_s(t) - R\,I(t) - L \frac{dI(t)}{dt} - \frac{1}{C}\int_{-\infty}^{t} d\tau\, I(\tau) = 0$

- Kirchhoff's current law: THE ALGEBRAIC SUM OF ALL BRANCH CURRENTS CONFLUENT IN THE SAME NODE IS ZERO

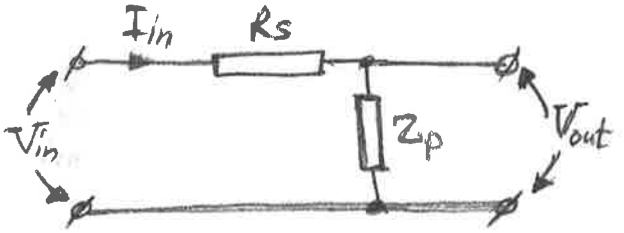
$$\sum I_K = 0$$

How it works:



$I_1 + I_2 + I_3 = 0$ ← no charge accumulation in the node!

Examples: a series - shunt circuit



Simple equations for phasors:

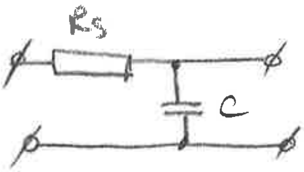
$$V_{out} = Z_p I_{in}$$
$$V_{in} = (R_s + Z_p) I_{in}$$

$$\Rightarrow \frac{V_{out}}{V_{in}} = \frac{Z_p}{R_s + Z_p} = \text{gain or attenuation}$$

It is convenient to express this in dB: $\left|\frac{V_{out}}{V_{in}}\right|(dB) = 20 \log_{10} \left|\frac{V_{out}}{V_{in}}\right|$

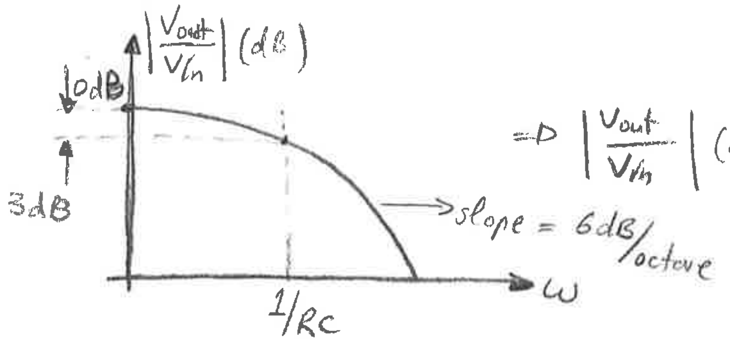
A few interesting cases:

a)



$$Z_p = \frac{1}{i\omega C} \Rightarrow \frac{V_{out}}{V_{in}} = \frac{1}{1 + i\omega C R_s}$$

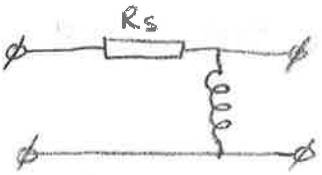
$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1 + \omega^2 C^2 R_s}}$$



$$\Rightarrow \left|\frac{V_{out}}{V_{in}}\right|(dB) = -10 \log_{10}\left[1 + \omega^2 C^2 R_s\right]$$

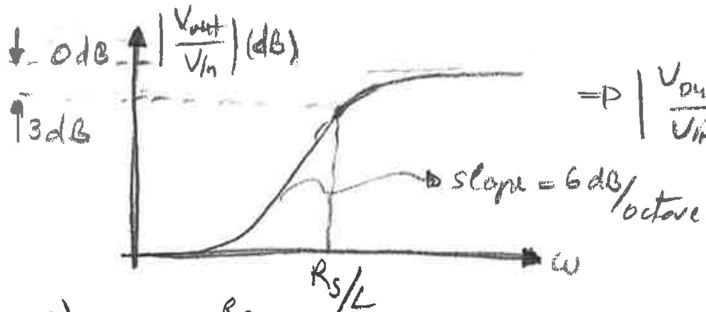
- works like a low-pass filter with cutoff $\sim 1/R_s C$

b)



$$Z_p = i\omega L \qquad \frac{V_{out}}{V_{in}} = \frac{1}{1 - \frac{iR_s}{\omega L}}$$

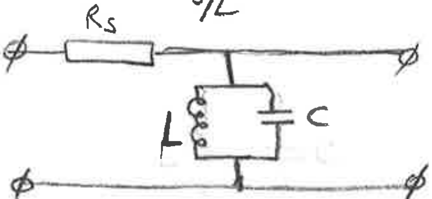
$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1 + \left(\frac{R_s}{\omega L}\right)^2}}$$



$$\Rightarrow \left|\frac{V_{out}}{V_{in}}\right|(dB) = -10 \log_{10}\left[1 + \left(R_s/\omega L\right)^2\right]$$

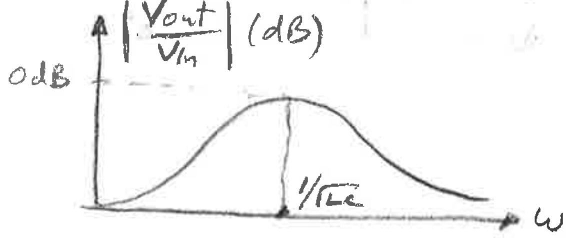
- works as a high-pass filter $\omega \gtrsim R_s/L$

c)



$$Z_p = \frac{i\omega L \cdot \frac{1}{i\omega C}}{i\omega L + \frac{1}{i\omega C}} = \frac{i\omega L}{1 - \omega^2 LC}$$

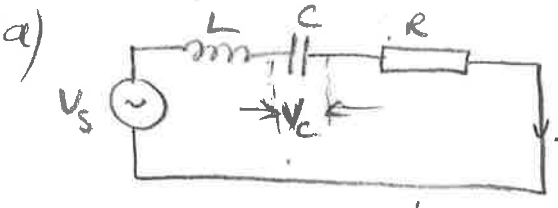
$$\frac{V_{out}}{V_{in}} = \frac{i\omega L}{R_s(1 - \omega^2 LC) + i\omega L}$$



$$\left|\frac{V_{out}}{V_{in}}\right|(dB) = 20 \log_{10} \frac{\omega L}{\sqrt{R_s(1 - \omega^2 LC)^2 + (\omega L)^2}}$$

Resonators based on lumped circuit elements

R L C components can be used to realize resonators.

a) 

Series RLC resonator

$$V_s(\omega) = Z(\omega)\cdot I(\omega)$$

$$Z(\omega) = R + iL\omega + \frac{1}{iC\omega}$$

$$= R + i\frac{L}{\omega}\left(\omega^2 - \omega_0^2\right) \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

resonance: $\omega = \omega_0$

$Z(\omega_0) = R$ → the impedance is real (resistive)

. the reactive part is zero, meaning that the inductor and capacitor reactances cancel each other.

Indeed

$$\begin{cases} \overline{W}_L = \frac{1}{4} L |I|^2 \\ \overline{W}_C = \frac{1}{4} C |V_C|^2 = \frac{1}{4}\frac{|I|^2}{C\omega^2} \end{cases}$$

but $|V_C| = \frac{|I|}{C\omega}$

so at resonance

$\omega = \omega_0 \Rightarrow \overline{W}_L = \overline{W}_C$

due to this, the energy oscillates between the capacitor and inductor and the source has to provide only what is lost through R.

- Quality factor

– suppose we put some energy $W(0)$ in the resonator. Due to the resistance R, this will be dissipated.

$$W(t) = W(0)\, e^{-\omega_0 t/Q}$$

Q = quality factor – it measures how well the resonator stores energy.

Now

$$-\frac{dW}{dt} = \frac{\omega_0 W}{Q}$$

$\overline{P}$ = average loss in a period

$$\overline{P} = \frac{1}{T}\int_0^T -\frac{dW}{dt}\,dt$$

$\frac{2\pi}{\omega_0} = T$ = period

or $\overline{P} = \frac{\omega_0}{Q}\underbrace{\frac{1}{T}\int_0^T W(t)\,dt}_{= W_T}$

W_T = total energy averaged over a period

So

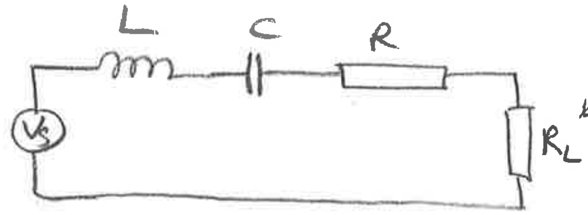
$$Q \equiv \frac{\omega_0 \overline{W}}{\overline{P}}$$

But $\begin{cases} \overline{W} = \overline{W_C} + \overline{W_L} = 2 \cdot \overline{W_L} = \dfrac{L\,|I|^2}{2} \\ \overline{P} = R \cdot \dfrac{|I|^2}{2} \end{cases}$

So

$$\boxed{Q = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}}$$

= Loading of a resonant circuit =



load resistor

In this case

$$Q_L = \left(\frac{1}{R} + \frac{1}{R_L}\right)\sqrt{\frac{L}{C}} = \text{loaded } Q$$

or $Q_L^{-1} = Q_{ext}^{-1} + Q^{-1}$

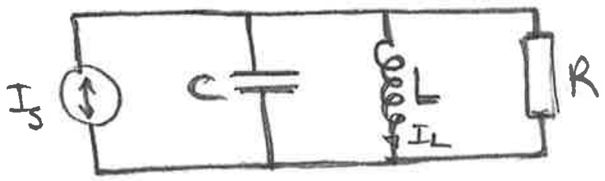
$$Q_{ext} = \frac{1}{R_L}\sqrt{\frac{L}{C}} = \text{external } Q$$

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \text{internal } Q$$

Simply because now

$$Q_L^{-1} = \frac{\overline{P}}{\omega_0 \overline{W}} = \frac{(R+R_L)\frac{|I|^2}{2}}{\frac{\omega_0 L\,|I|^2}{2}} = \frac{R}{\omega_0 L} + \frac{R_L}{\omega_0 L}$$

b)



parallel RLC resonator

$$Z(\omega) = \frac{1}{G + i\frac{C}{\omega}(\omega^2 - \omega_0^2)} = \frac{1}{Y(\omega)}$$

$$G = 1/R$$

A similar idea: $\overline{W} = \overline{W_C} + \overline{W_L} = \frac{1}{4}|V|^2 \cdot C + \frac{1}{4} L |I_L|^2$

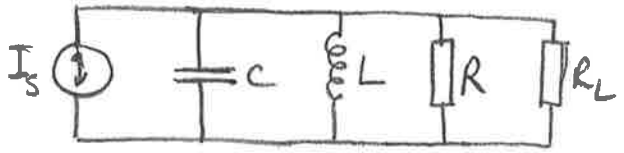
But $I_L = \dfrac{V}{i\omega L}$

$= \frac{1}{2} C |V|^2$ at resonance $\omega = \omega_0 = \frac{1}{\sqrt{LC}}$

$$P_T = \frac{1}{2} G |V|^2$$

$$\Rightarrow Q = \frac{\omega_0 \overline{W}}{\overline{P}} = \frac{\omega_0 C}{G} = \omega_0 R C = \frac{R}{\omega_0 L} = R\sqrt{\frac{C}{L}}$$

— Loading of the parallel RLC resonator —



$Q_L = \left(\frac{1}{R} + \frac{1}{R_L}\right)\sqrt{\frac{C}{L}}$ = loaded Q

$Q_{ext} = \frac{1}{R_L}\sqrt{\frac{C}{L}}$ = external Q

$Q = \frac{1}{R}\sqrt{\frac{C}{L}}$ = internal Q

or $Q_L^{-1} = Q_{ext}^{-1} + Q^{-1}$ → this relation is the same as for the series RLC resonator