ELEC-C8201: Control Theory and Automation

Exercise 8 - Solutions

The problems marked with an asterisk (⋆) are not discussed during the exercise session. The solutions are given in MyCourses and these problems belong to the course material.

1. Consider the *single-input, single-output* system described by

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t),$$
$$y(t) = C\mathbf{x}(t)$$

Determine whether the system is controllable and observable when:

a) $A = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 2 \end{bmatrix}.$

b) $A = \begin{bmatrix} -10 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$

c) $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$

Solution.

a) Controllability:

$$P_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix} \Rightarrow \det(P_c) = \begin{vmatrix} 0 & 1 \\ 1 & -3 \end{vmatrix} = -1 \neq 0 \quad (\rightarrow \text{controllable})$$

Observability:

$$P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & -6 \end{bmatrix} \Rightarrow \det(P_o) = 0 \quad (\rightarrow \text{not observable})$$

b) Controllability:

$$P_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix} \Rightarrow \det(P_c) = 0 \quad (\rightarrow \text{not controllable})$$

Observability:

$$P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -10 & 0 \end{bmatrix} \Rightarrow \det(P_o) = 0 \quad (\rightarrow \text{not observable})$$

Note that in this case, the input is only on x_2 and therefore x_1 cannot be controlled. Given, however, that A is diagonal, we can easily see that x_1 goes to zero for whatever input and hence the system is **stabilizable**. Similarly, If state x_2 goes to zero, the system is **detectable**.

c) Controllability:

$$P_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix} \Rightarrow \det(P_c) = \begin{vmatrix} -1 & 2 \\ 2 & -3 \end{vmatrix} = 3 - 4 = -1 \neq 0 \quad (\rightarrow \text{controllable})$$

Observability:

$$P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \det(P_o) = 1 \quad (\rightarrow \text{observable})$$

2. Consider the second-order system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(t)$$

For what values of k_1 and k_2 is the system completely controllable?

Solution.

$$P_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} k_1 & k_1 - k_2 \\ k_2 & -k_1 + k_2 \end{bmatrix}$$

Therefore, the determinant is given by

$$\det(P_c) = \begin{vmatrix} k_1 & k_1 - k_2 \\ k_2 & -k_1 + k_2 \end{vmatrix} = -k_1(k_1 - k_2) - k_2(k_1 - k_2) = -(k_1 - k_2)(k_1 + k_2)$$

So, the system is completely controllable if and only if $|k_1| \neq |k_2|$ (or $k_1^2 \neq k_2^2$).

3. Consider the third-order system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -5 & -3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 2 & -4 & 0 \end{bmatrix} \mathbf{x}(t)$$

a) Verify that the system is observable.

b) Determine the observer gain matrix required to place the observer poles at $s_{1,2} = -1 \pm j$ and $s_3 = -5$.

Solution.

a) Observability:

$$P_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 2 & -4 & 0 \\ 0 & 2 & -4 \\ 32 & 20 & 14 \end{bmatrix} \Rightarrow \det(P_o) = 728 \neq 0 \quad (\rightarrow \text{observable})$$

b) The desired characteristic equation is given by

$$\begin{aligned} p(s) &= [s-(-1+j)][s-(-1-j)](s+5) \\ &= [(s+1)^2+1](s+5) \\ &= (s^2+2s+2)(s+5) \\ &= s^3+7s^2+12s+10 \end{aligned}$$

1st way: Using the Ackermann's formula for the observer gain matrix,

$$L = p(A)P_o^{-1}\begin{bmatrix} 0 & 0 & \dots & 1 \end{bmatrix}^T = \dots = \begin{bmatrix} 0.14 \\ -0.93 \\ 0.79 \end{bmatrix}$$

where $p(A)$ is given by

$$p(A) = A^3 + 7A^2 + 12A + 10I.$$

2nd way: Using the determinant:

$$|sI-(A-LC)| = \left| \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -5 & -3 \end{bmatrix} + \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} 2 & -4 & 0 \end{bmatrix} \right|$$

4. Suppose that the vector differential equation describing the inverted pendulum is given by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 9.8 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} u(t)$$

Assume that all state variables are available for measurement and use state variable feedback. Place the system characteristic roots at $s - -2 \pm j$,-5, and -5.

Solution. Consider the state variable feedback law $u = K\mathbf{x}$. Using the Ackermann's formula

$$K = \begin{bmatrix} -14.2045 & -17.0455 & -94.0045 & -31.0455 \end{bmatrix}$$

5. A system is represented by the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \frac{du}{dt} + u$$

where $y =$ output and $u =$ intput.

a) Develop a state variable representation and show that it is a controllable system.

b) Define the state variables as $x_1 = y$ and $x_2 = dy/dt - u$, and determine whether the system is controllable.

Solution.

a) In the first case:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore,

$$P_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \Rightarrow \det(P_c) = \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1 \neq 0 \quad (\rightarrow \text{controllable})$$

b) In the second case:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Therefore,

$$P_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \Rightarrow \det(P_c) = \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 1 - 1 = 0 \quad (\rightarrow \text{not controllable})$$

Note: the controllability of a system depends on the definition of the state variables!

6. Consider the second-order system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 1 & 2 \\ -6 & 12 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -5 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 4 & -3 \end{bmatrix} \mathbf{x}(t)$$

a) Verify that the system is observable and controllable.

b) Design a full-state feedback law and an observer by placing the closed-loop system poles at $s_{1,2} = -1 \pm j$ and the observer poles at $s_{1,2} = 12$.

Solution.

a) Controllability:

$$P_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ 1 & 18 \end{bmatrix} \Rightarrow \det(P_c) = -87 \neq 0 \quad (\rightarrow \text{controllable})$$

Observability:

$$P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 22 & 44 \end{bmatrix} \Rightarrow \det(P_o) = 242 \neq 0 \quad (\rightarrow \text{observable})$$

b) The controller gain matrix $K = \begin{bmatrix} 3.02 & 6.11 \end{bmatrix}$ places the closed-loop system poles at $s_{1,2} = -1 \pm j$, and the observer gain matrix $L = \begin{bmatrix} 2.38 & -1.16 \end{bmatrix}^T$ observer poles at $s_{1,2} = 12$.